Partial Differential Equation

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Abstract

This document presents notes from Partial Differential Equation.

This note is dedicated to Professor Ovidiu Savin.

Contents

1 Partial Differential Equation **4**
1.1 Introduction . . . 4
1.2 Solve Variable Coefficient Equation . . . 4
1.2.1 Burger's Equation . . . 5
1.3 Flows, Vibrations, and Diffusions . . . 6
1.3.1 Transport . . . 6
1.3.2 Vibrating String . . . 6
1.3.3 Vibrating Drumhead . . . 7
1.4 Second-Order PDE . . . 7

2 Laplace Equation **9**

3 Wave Equation **9**

4 Heat Equation **16**
4.1 Introduction . . . 16
4.2 An Example . . . 17
4.3 Uniqueness . . . 17
4.4 Convolution . . . 19
4.5 An Application: Random Walk . . . 21

5 Schrodinger Equation **21**

6 Fourier Series **21**
6.1 Newmann Boundary . . . 23
6.2 Robin Conditions . . . 24
6.3 Fourier in Wave Equation . . . 25
6.4 Riemann-Lebesgue-Fourier . . . 28

7 Harmonic Functions **34**
7.1 Harmonic Functions: Laplace Equation . . . 34
7.2 Solve Dirichlet Problem . . . 39
7.3 More Dirichlet Problems . . . 43
7.4 Mean Value Property . . . 44
7.5 Dirichlet Principle . . . 47

8 GREEN'S FUNCTION **48**
8.1 Fundamental Solution . . . 50
8.2 Green's Function Solving Dirichlet Problem . . . 50
8.3 Half-space in Sphere . . . 51

1 Partial Differential Equation

1.1 Introduction

The key defining property of a partial differential equation (PDE) is that there is more than one independent variable $x, y, ...$ There is a dependent variable that is an unknown function of these variables $u(x, y, ...)$. We will denote its derivatives by subscripts, that is, $\partial u/\partial x = u_x$. A PDE is an identity that relates the independent variables, the dependent variable u, and the partial derivatives of u. It can be written as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0$$

Some notable examples are the following:
1. $u_x + u_y = 0$ (transport)
2. $u_x + yu_y = 0$ (transport)
3. $u_x + uu_y = 0$ (shock wave)
4. $u_{xx} + u_{yy} = 0$ (Laplace's equation)
5. $u_{tt} - u_{xx} + u^3 = 0$ (wave with interaction)
6. $u_t + uu_x + u_{xxx} = 0$ (dispersive wave)
7. $u_{tt} + u_{xxxx} = 0$ (vibrating bar)
8. $u_t - iu_{xx} = 0$ ($i = \sqrt{-1}$) (quantum mechanics)

1.2 Solve Variable Coefficient Equation

In 2D case, we have $a(x, y)u_x + b(x, y)u_y = c(x, y)$, we have $x' = a(x, y)$, $y' = b(x, y)$, and $z' = c(x, y)$ while $d/dtu(x(t), y(t)) = c(x, y)$. Given initial condition $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$. If $u(x_0, y_0) = z_0$, we can determine u along the trajectory of the ODE.

Remark 1.2.1. If $c(x, y)$ is replaced by $x(x, y, u)$, then ODE is $x' = a(x, y)$, $y' = b(x, y)$, and $z' = c(x, y, z)$.

Example 1.2.2. Consider $u_x + yu_y = 0$.

Example 1.2.3. Consider $u_x + 2xy^2u_y = 0$.

Let us recall transport equation. Consider $u(t, x) = u(t, x_1, x_2, ..., x_n)$ while $x \in \mathbb{R}^n$. We also know

$$u_t + \sum_{i=1}^{n} a_i(t, x)u_{x_i} = 0$$

while initial condition is $u(0, x) = g(x)$ is a given function. We solve the ODE with $x' = a(t, x)$ and $z' = f(t, x)$ while $d/dtu(t, x(t)) = f(t, x)$ which is the function above. At $t = 0$, $x(0) = x_0$ and $z(0) = g(x_0)$ would be the initial data.

Example 1.2.4. Consider $u(t, x, y)$ while $(x, y) \in \mathbb{R}^2$ and also know

$$u_t - yu_x + xu_y = u$$

which is the transport equation. Also we know some initial data $u(0, x, y) = x^2 + y^2$. The goal is to solve u uniquely.

Note characteristic ODE is $x' = -y$, $y' = x$, and $z' = z$. Then we have $(x(t), y(t)) = (x_0, y_0) + n(\cos t, \sin t)$ while $n = \sqrt{X^2 + Y^2}$. Then we know $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$. We know how to solve z and $z(t) = z_0e^t$. The trajectory is a swirl, like a tornado, and can be described as

$$u(t, x, y) = (x^2 + y^2)e^t$$

Now let us take a moment to discuss non-linear first order PDE (may not be essence for this course, but good tools to have). Consider

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

to be our equation. Then we are looking at graph of u,

$$G_u = \{z = u(x,y)\}$$

which gives us some sort of surface in a 3D coordinate. Given initial or a precise point, say $u(x_0, y_0, z_0)$. We can write

$$(-u_x, -u_y, 1) \cdot (a,b,c) = 0$$

which tells us at point z the vector (a,b,c) has to be tangent to the plane, that is, perpendicular to the directional vector. Now the characteristic curves are given by the ODE, $x' = a(x,y,z)$, $y' = b(x,y,z)$, and $z' = c(x,y,z)$.

Example 1.2.5. Burger's Equation. Consider $u_t + uu_x = 0$ and $u(0,x) = g(x)$ given. Next, we have $u(t, x+tg(x)) = g(x)$ and the derivative $u_x(1+tg') = g'$ which implies $u_x = \frac{g'}{1+tg'}$

1.2.1 Burger's Equation

Burgers' Equation is a fundamental PDE occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, traffic flow. For a given field, $y(x,t)$ and diffusion coefficient d, the general form of Burgers' equation in one space dimension is the dissipative system:

$$\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} = d\frac{\partial^2 y}{\partial x^2}$$

Added space-time noise $\eta(x,t)$ forms a stochastic Burgers' equation

$$\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} = d\frac{\partial^2 y}{\partial x^2} = d\frac{\partial^2 y}{\partial x^2} - \lambda\frac{\partial \eta}{\partial x}$$

This stochastic PDE is equivalent to the Kardar - Parisi - Zhang equation in a field $h(x,t)$ upon substituting

$$y(x,t) = -\lambda \partial h/\partial x$$

But whereas Burgers' equation only applies in one spatial dimension, the Kardar - Parisi - Zhang equation generalizes to multiple dimensions. When the diffusion term is absent (i.e. $d = 0$), Burgers' equation becomes the inviscid Burgers' equation:

$$\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} = 0,$$

which is a prototype for conservation equations that can develop discontinuities.

Remark 1.2.6. Kardar-Parisi-Zhang Equation. It is a non-linear stochastic partial differential equation. It describes change of the height $h(\overrightarrow{x}, t)$ at place $\overrightarrow{x}$ and time t. Formally, it is stated

$$\frac{\partial h(\overrightarrow{x}, t)}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\overrightarrow{x}, t),$$

where $\eta(\overrightarrow{x}, t)$ is white Gaussian noise with average $< \eta(\overrightarrow{x}, t) >= 0$ and second moment

$$< \eta(\overrightarrow{x}, t)\eta(\overrightarrow{x}', t) >= 2D\delta^d(\overrightarrow{x} - \overrightarrow{x}')\delta(t - t')$$

while ν, λ, and D are parameters of the model and d is the dimension.

Remark 1.2.7. Burgers' Equation. It is a fundamental PDE occurring in various areas of applied mathematics, such as fluid mechanics, non-linear acoustics, gas dynamics, traffic flow.

For a given field $y(x,t)$ and diffusion coefficient d, the general form of Burgers' equation (also known as viscous Burgers' equation) in one space dimension is the dissipitive system:

$$\frac{\partial y}{\partial t} + y\frac{y}{\partial x} = d\frac{\partial^2 y}{\partial x^2}$$

Added space-time noise $\eta(x,t)$ forms a stochastic Burgers' equation

$$\frac{\partial y}{\partial t} + y\frac{\partial y}{\partial x} = d\frac{\partial^2 y}{\partial x^2} - \lambda\frac{\partial \eta}{\partial x}$$

1.3 Flows, Vibrations, and Diffusions

1.3.1 Transport

Consider a fluid, water, say, flowing at a constant rate c along a horizontal pipe of fixed cross section in the positive x direction. A substance, say a pollutant, is suspended in the water. Let $u(x,t)$ be its concentration in grams/centimeter at time t. Then

$$u_t + cu - x = 0$$

The amount of pollutant in the interval $[0,b]$ at the time t is $M = \int_0^b u(x,t)dx$, in grams, say. At the later time $t+h$, the same molecules of pollutant have moved to the right by $c \cdot h$ centimeters. Hence,

$$M = \int_0^b u(x,t)dx = \int_{ch}^{b+ch} u(x,t+h)dx$$

Differentiating with respect to b, we get

$$u(b,t) = u(b+ch, t+h)$$

Differentiating with respect to h and putting $h = 0$, we get

$$0 = cu_x(b,t) + u_t(b,t)$$

which is equation $u_t + cu_x = 0$ in the beginning of this sub section.

1.3.2 Vibrating String

Consider a flexible, elastic homogeneous string or thread of length l, which undergoes relatively small transverse vibrations. Newton's law $F = ma$ in its longitudinal x and transverse u components is

$$\left.\frac{T}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} = 0 \text{ (longitudinal)}$$

$$\left.\frac{Tu_x}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt} dx \text{ (transverse)}$$

The right sides are the components of the mass times the acceleration integrated over the piece of string. Since we have assumed that the motion is purely transverse, there is no longitudinal motion. The second equation from above, differentiated, says that

$$(Tu_x)_x = \rho u_{tt}$$

That is,

$$u_{tt} = c^2 u_{xx} \text{ where} c = \sqrt{\frac{T}{\rho}}$$

which is called the wave equation with c the wave speed.

There are many variations of this equation:

(i) If significant air resistance r is present, we have an extra term proportional to the speed u_t, thus:

$$u_{tt} - c^2 u_{xx} + r u_t = 0 \text{ where } r > 0.$$

(ii) If there is a transverse elastic force, we have an extra term proportional to the displacement u, as in a coiled spring, thus:

$$u_{tt} - c^2 u_{xx} + ku = 0 \text{ where } k > 0.$$

(iii) If there is an externally applied force, it appears as an extra term, thus:

$$u_{tt} - c^2 u_{xx} = f(x,t)$$

which makes the equation inhomogeneous.

1.3.3 Vibrating Drumhead

The two-dimensional version of a string is an elastic, flexible, homogeneous drumhead, that is, a membrane stretched over a frame. Say the frame lies in the xy plane, $u(x,y,t)$ is the vertical displacement, and there is no horizontal motion. The horizontal components of Newton's law again give constant tension T. Let D be any domain in the xy plane, say a circle or a rectangle. Let bdy D be its boundary curve. We use reasoning similar to the one-dimensional case. The vertical component gives (approximately)

$$F = \int_{\text{bdy} D} T \frac{\partial u}{\partial n} ds = \iint_D \rho u_{tt} dx dy = ma,$$

where the left side is the total force acting on the piece D of the membrane, and where $\partial u / \partial n = n \cdot \bigtriangledown u$ is the directional derivative in the outward normal direction, n being the unit outward normal vector on bdy D. By Green's theorem, this can be rewritten as

$$\iint_D \bigtriangledown \cdot (T \bigtriangledown u) dx dy = \iint_D \rho u_{tt} dx dy.$$

Since D is arbitrary, we deduce from the second vanishing theorem that $\rho u_{tt} = \bigtriangledown \cdot (T \bigtriangledown u)$. Since T is constant, we get

$$u_{tt} = c^2 \bigtriangledown \cdot (\bigtriangledown u) \equiv c^2 (u_{xx} + u_{yy})$$

where $c = \sqrt{T/\rho}$ as before and $\bigtriangledown \cdot (\bigtriangledown u) = \text{div grad } u = u_{xx} + u_{yy}$ is known as the two-dimensional laplacian.

1.4 Second-Order PDE

Recall the notion of first-order PDE, say $u(x,y)$ that is non-linear. Let

$$a(x,y,u) u_x + b(x,y,u) u_y = c(x,y,u)$$

we solve it using characteristic ODE. That is, we look at

$$x' = a(x,y,z), y' = b(x,y,z), \text{ and } z' = c(x,y,z)$$

which are ODEs. Consider initial condition $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$. If, given, $u(x_0, y_0) = z_0$, then $u(x(t), y(t)) = z(t)$ for all t. What if we are given function of t and x, i.e. $u(t, x)$, that is

$$u_t + uu_x = 0$$

with initial data $u(0, x) = \varphi_0(x)$. How to solve it? We use s for the variable in ODE and we want to simplify this to an ODE. Let

$$dt/ds = 1, dx/ds = z, \text{ and } dz/ds = 0$$

with initial data $t(0) = 0$, $x(0) = x_0$, and $z(0) = \varphi(x_0)$.

To solve this, note that $z = \varphi(x_0)$ which is where z starts with from data. Also known that $t = s$ since the rates of change for them are the same. Moreover, we have $x(s) = x_0 + s\varphi(x_0)$. Hence, now we have $(t, x_0 + \varphi(x_0)t, \varphi(x_0)$ which looks like a line with positive slope if plotted on 2D frame (x and t). However, in 3D (x, t, z), the characteristic curve is a line goes up. That is, we are looking at

$$u(t, x_0 + \varphi(x), t) = \varphi(x_0)$$

To find the u at (t, x), we need to solve the equation $x = x_0 + t\varphi(x_0)$ for x_0.

Example 1.4.1. Consider $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t)$ as our premise, which can be rewritten as $Au = \sin(t + x)$ where operator A is

$$A = 3\frac{\partial^2}{\partial t^2} + 10\frac{\partial^2}{\partial t\partial x} + 3\frac{\partial^2}{\partial x^2}$$

while, in this case, the operator can be rearranged and be simplified to

$$A = \left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right)\left(3\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$$

Now we change the variables so we have two individual partial derivative.

$$\frac{\partial}{\partial y} = \left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right), \text{ and } \frac{\partial}{\partial z} = \left(3\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$$

which will lead us to the following change of variables

$$t = y + 3z, \text{ and } x = 3y + z$$

and also define $w(y, z) = u(t, x)$. Then we have

$$\frac{\partial^2 w}{\partial y\partial z} = Au = \sin(x + t) = \sin(4y + z)$$

Integrate twice on both sides we will get the answer,

$$\iint \frac{\partial^2 w}{\partial y\partial z} = \iint \sin(4y + z)$$

Consider integral over an region:

Example 1.4.2. Given

$$\iint_R u_t + uu_x dA = \int_T u dx - \int_B u dx$$

note that left side is a full derivative $uu_x = ((1/2)u^2)_x$. Thus,

$$\iint_R u_t + uu_x dA = \int_T u dx - \int_B u dx = \int_R \frac{1}{2}u^2 dt - \int_L \frac{1}{2}u^2 dt$$

From §14.1 Speed of Discontinuity in text [1], we discuss Rankine–Hugoniot formula, which is conditioned by Rankine–Hugoniot conditions.

Example 1.4.3. Consider $u_t + uu_x = 0$ for $t \geq 0$; and $u(0,x) = \varphi_0(x)$. Then we can solve for $u(t, w + t\varphi(w)) = \varphi(w)$. For large value of t, the dominated function $x = w + t\varphi(w)$.

Let us discuss four main examples of second order PDE's.

2 Laplace Equation

Consider $u(x,y)$ with constant coefficients, a simple form, as the following: $u(x,y)$. Given $u_{xx} + u_{yy} = 0$ which is called Laplace equation. Also in some other sources scholars may cite $\triangle u$ as the Laplace of u.

We can also write $u_{xx} + u - yy = \partial_x u_x + \partial_y u_y = \text{div}\nabla u$. Recall Green's Theorem, $0 = \iint_D \text{div} F dxdy = \int_{\partial D} F \cdot \nu de$. Then in this case we have $\text{div}(\nabla u) = 0$, the flux of ∇u through the boundary of any domain is 0. At the point (x,y), we can discuss the Hessian of u at the point (x,y): that is a symmetric matrix

$$D^2 u(x,y) = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix}$$

Then u can be expressed as linear part plus extra terms:

$$u(x,y) = \underbrace{u(x_0,y_0) + \nabla(x_0,y_0) \cdot ((x,y) - (x_0,y_0))}_{\text{linear term}} + \underbrace{\frac{\lambda_1}{2}(x-x_0)^2 + \frac{\lambda_2}{2}(y-y_0)^2}_{\text{quadratic part}}$$

$$+0 \cdot (|(x,y) - (x_0,y_0)|^3)$$

Moreover, we can discuss the boundary conditions. We notate $u_{xx} + u_{yy} = \triangle u$. Then we have $\triangle u = 0$ in domain D; and also $u = \varphi$ given on ∂D. Now we want to find u. We expect u to be determined uniquely from the given data; and to have continuous dependence with respect to the data. Such expectation is well posed of the problem with the boundary condition.

Other boundary conditions can be: Newmann Problem. Given $\triangle u = f(x,y)$ given in D and $u_\nu = \psi$ given on ∂D.

One can also do this in higher dimensions. Given $u_{x_1,x_1} + u_{x_2,x_2} + \cdots + u_{x_n,x_n} = 0$ which is $\triangle u = \text{div}(\nabla u) = u_{x_1,x_1} + \ldots u_{x_n,x_n}$.

3 Wave Equation

Consider $u_{xx} - u_{yy} = 0$ which is known as the wave equation. Moreover, we can have

$$u_{tt} = c^2 u_{xx}$$

with the right-hand side being the Face. Naturally, we have boundary conditions for $u(x,t)$ for $x \in \mathbb{R}$, $t \in [0,\infty]$. to be the following

$$u_{tt} = c^2 u_{xx} \text{ in } \mathbb{R} \times (0,\infty)$$

$$u(x,0) = \varphi(x)$$
$$u_t(x,0) = \psi(x)$$

with infinite string.

What if we have something that is bounded? Let us consider the following domain $x \in 0, L]$ and $t \in [0, \infty]$. Then given

$$u_{tt} = c^2 u_{xx} \text{ in } (0, L) \times (0, \infty)$$

$$u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x), u(0, t) = h(t), u(L, t) = \rho(t)$$

Next, we can move on to higher dimensions. Consider $u_{tt} = u_{xx}$. Then we can consider $u_{tt} = c^2(u_{x_1x_1} + u_{x_2x_2})$ with $u_{tt} = c^2 \triangle u$. Consider constant coefficient linear second order equations on two-dimensional $u(x_1, x_2)$ solves

$$a_{11}u_{x_1x_1} + 2a_{12}u_{x_1x_2} + a_{22}u_{x_2x_2} + a_1u_{x_1} + a_2u_{x_2} + a_0u = 0$$

and we have

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

and

$$D^2u = \begin{pmatrix} u_{x_1x_1} & u_{x_1x_2} \\ u_{x_1x_2} & u_{x_2x_2} \end{pmatrix}$$

this gives us $\vec{a} = (a_1, a_2)$ and $\nabla u = (u_{x_1}, u_{x_2})$, which gives

$$t_n(AD^2u) + \vec{a} \cdot \nabla u + a_0u = 0$$

Recall $x = (x_1, x_2)$ and $y = (y_1, y_2)$ for $u(y_1, y_2)$ and by chain rule

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u}{\partial y_2}\frac{\partial y_2}{\partial x_1}$$

$$u_{x_1} = u_{y_1}\frac{\partial y_1}{\partial x_1} + u_{y_2}\frac{\partial y_2}{\partial x_1}$$

$$u_{x_2} = u_{y_1}\frac{\partial y_1}{\partial x_2} + u_{y_2}\frac{\partial y_2}{\partial x_1}$$

and thus

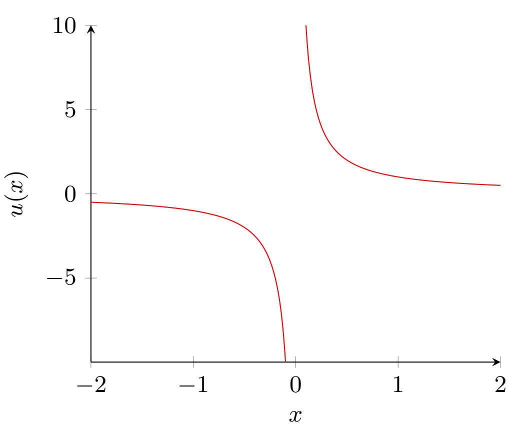
$$\nabla_x u = \nabla_y u \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix}$$

while the right-hand side is $(u_{y_1}, u_{y_2})D_x y$.

Example 3.0.1. Consider $yu_{xx} - 2u_{xy} + xu_{yy} = 0$ and we consider matrix

$$\begin{pmatrix} y & -1 \\ -1 & y \end{pmatrix}, \text{ and } \det = yx - 1$$

hence we are looking at elliptic hyperbolic curve.



Let us further this topic to discuss one-dimensional wave equation. Consider $u_{tt} = u_{xx}$. We can assume $c = 1$ by rescaling time. Attempt to write $u(x,t) = \tilde{u}(x, ct)$ while we define $\tilde{u}$ as

$$u(x,t) = \tilde{u}(x, ct)$$

$$u_x = \tilde{u}_x, u_{xx}\tilde{u}_{xx}$$

By Chain rule, $u_t = c\tilde{u}_t$ and $u_{tt} = c^2\tilde{u}_{tt}$ which gives us

$$\tilde{u}_{tt} = \tilde{u}_{xx}$$

Then the general formula for

$$u_{tt} = u_{xx}$$

and we solve it to get D'Alambert formla. Referring to page 36 of text [1]. To stress this point, we start with wave equation

$$u_{tt} = c^2 u_{xx}, \text{ for } -\infty < x < +\infty$$

This is simplest second-order equation. The reason is that the operator factors nicely:

$$u_{tt} - c^2 u_{xx} = (\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})u = 0$$

This means that starting from a function $u(x,t)$, you compute $u_t + cu_x$, call the result ν, then you compute $\nu_t - c\nu_x$, and you ought to get the zero function. The general solution is

$$u(x,t) = f(x+ct) + g(x-ct)$$

where f and g are two arbitrary functions of a single variable. From theorem, we are given general equation

$$u(x,t) = f(x+ct) + g(x-ct)$$

and we take derivatives to obtain

$$u'(x,t) = cf'(x+ct) + cg'(x-ct)$$

Next, we set up the system. For $t = 0$, we obtain

$$u(x,0) = f(x) + g(x) = \phi(x)$$

$$u_t(x,0) = cf'(x) - cg'(x) = \psi(x)$$

This gives us

$$2f'(x) = \phi(x) + \frac{1}{c}\psi(x) \Rightarrow f'(x) = \frac{1}{2}\phi'(x) + \frac{1}{2c}\psi(x)$$

and

$$g'(x) = \phi'(x) - f'(x) = \frac{1}{2}\phi'(x) - \frac{1}{2c}\psi(x)$$

which we can integrate both sides to obtain

$$\int f' = \frac{1}{2}\int \phi' + \frac{1}{c}\psi \Rightarrow f = \frac{1}{2}\phi + \frac{1}{2c}\int \psi + A$$

$$\int g' = \frac{1}{2}\int \phi' - \frac{1}{c}\psi \Rightarrow g = \frac{1}{2}\phi - \frac{1}{2c}\int \psi + B$$

which serves as expressions of f and g for us to substitute them into $u = f + g$, thus,

$$u(x,t) = f(x+ct) + g(x-ct) = \frac{1}{2}[\phi(x+ct) - \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds$$

which is the famous d'Alembert equation.

Example 3.0.2. The plucked string.

Consider $u_{tt} = u_{xx}$ and we have $\varphi(x) = 1 - |x|$ while $x \in [-1, 1]$ and $\varphi(x) = 0$ while $x \notin [-1, 1]$. More, we have $u_t(x, 0) = \psi(x) = 0$. This gives us

$$u(x,t) = \frac{1}{2}(\varphi(x+t) + q(x-t))$$

Consider 1-D wave equation

$$u_{tt} = u_{xx} \text{ for } x \in \mathbb{R}, t \in [0, \infty)$$

with at $t = 0$ the function has initial condition $u = \varphi(x)$ and $u_t = \psi(x)$. Then

$$u(x,t) = \frac{1}{2}(\varphi(x+t) + \varphi(x-t)) + \frac{1}{2}\int_{x-t}^{x-t} \psi(s) ds$$

However, for wave equation with source

$$u_{tt} = u_{xx} + f(x,t)$$

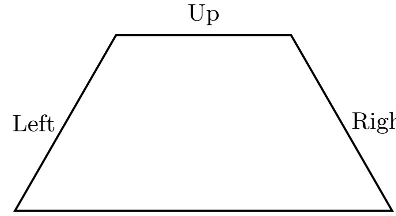
and in this case say initial condition is at $t = 0$ $u = \varphi$ and $u_t = \varphi$. Then we have

$$(\partial_t - \partial_x)(\partial_t + \partial_x)u = u_{tt} - u_{xx} = f(x,t)$$

Recall last time, we have $u_{tt} = u_{xx}$ and we want

$$\iint_D (u_{tt} - u_{xx}) u_t dx dt = \int_{\partial D} \frac{1}{2} u_t^2 \nu_t + \frac{1}{2} \mu_x^2 \nu_t - u_x u_t \nu_x ds$$

this is just like



Next, we get

$$0 = \int_T \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 ds - \int_B \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 ds + \int_R \frac{1}{2\sqrt{2}} (u_t - u_x)^2 ds + \int_L \frac{1}{2\sqrt{2}} (u_t + u_x)^2 ds$$

Hence, we conclude (1) and (2) respectively,

$$\int_B \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 \geq \int_T \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 ds$$

and

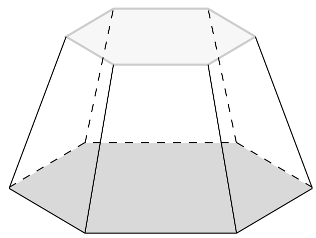
$$\text{If } u \equiv 0 \text{ on } R \text{ and } L, \text{ then } E(B) = E(T)$$

Assume that initial data φ and ψ vanish outside some large interval. Then $E(t) = \int \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 dx$ is constant in t.

Now let us discuss 3-D wave equation:

$$u_{tt} = \triangle u$$

while $u(x_1, x_2, x_3, t) = u(x, t)$ with $x \in \mathbb{R}^3$. The Laplace $\triangle u = u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3}$, which is the summation of three derivatives. This looks like the graph below



Consider the graph above, we have the surface inside to be D and we know that the vector $\nu = (\nu_x, \nu_t)$ while $\nu_x \in \mathbb{R}^3$ and $\nu_t \in \mathbb{R}$. Integrating with Green Theorem,

$$\begin{aligned} 0 &= \iint_D (u_{tt} - \triangle u) u_t dx dt \\ &= \int_{\partial D} \frac{1}{2} u_t^2 \nu_t + \sum_{i=1}^{3} \frac{1}{2} u_{x_i}^2 \nu_t - u_t \sum_{i=1}^{3} u_{x_i} \nu_{x_i}, \ [1] \\ &= -\int_B \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla_x u|^2 d\sigma + \int_T \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla_x u|^2 d\sigma \\ &= \int_C \nu_t [\frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 - u_t (\nabla u \cdot \frac{\nu_x}{\nu_t})] d\sigma \end{aligned}$$

note that to do [1], we need 1-D computation and replace ν_x by ν_{x_i} and sum over i.

Finally, we can finally conclude

$$\int_B \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla_x u|^2 d\sigma \geq \int_T \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla_x u|^2 d\sigma$$

The work is done in text page 232 [1]. We can also refer to the same place for computation in 3-D.

Let us consider another example. Given

$$u_{tt} = \triangle u, t > 0, u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x)$$

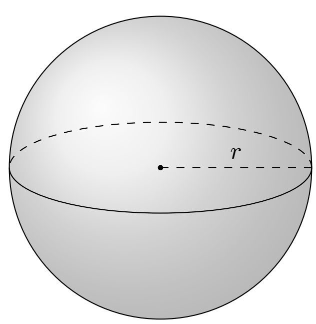
For notation, denote the following

$$S_n(x) = \{y \in \mathbb{R}^3 : |y - x| = n\}$$

which is the sphere of center x and radius r. Moreover, denote the average

$$\fint_{S_r(x)} v(y) dS_y = \frac{1}{\text{area}(S_r(x))} \int_{S_r(x)} v(y) dS_y$$

which is a sphere with radius r and centered at x.



Now the trick is to reduce the 3D wave equation to 1D wave equation.

$$\bar{u}(x,t) = \int_{S_{|x|}(0)} u(y,t) dS_y$$

and we assert that $\bar{u}$ solves wave equation but it is a radial function. That is, $\bar{u}(x,t) = v(|x|,t) = v(r,t)$, and $\bar{u}(0,t) = u(0,t)$. Thus, we have

$$\bar{u}_t t = \triangle \bar{u}, \bar{u}(x,0) = \bar{\varphi}(x) = \int_{S_r(0)} \varphi dS$$

$$\bar{u}_t(x,0) = \bar{\psi}(x) = \int_{S_r(0)}^{S_r(0)} \psi dS$$

Then

$$\bar{u}_{tt} = v_{tt}$$

$$\triangle \bar{u} = v_{rr} + \frac{n-1}{r} v_r$$

The equation for v is

$$v_{tt} = v_{rr} + \frac{2}{n} v_r$$

$$v(r,0) = \int_{S_n(0)} \varphi$$

$$v_t(r,0) = \int_{S_r(0)} \psi$$

The claim is that $v_{tt} = v_{rr} + (2/n)v_r \to (rv)_{tt} = (rv)_{rr}$. Then $rv_{tt} = (v + rv_r)_r = v_r + v_r + rv_{rr}$. This gives us d'Alembert formula,

$$(rv)(r,t) = \frac{1}{2}(\bar{\varphi}(r+t) - \bar{\psi}(t-r)) + \frac{1}{2}\int_{t-n}^{t+n} \bar{\psi}(s) dS$$

while $rv = \bar{\varphi}$ and $(rv)_t = \bar{\psi}$. Divide by n and let $n \to 0$ we have

$$v(0,t) = (\bar{\varphi})'(t) + \bar{\psi}(t)$$

Hence, we conclude

$$u(0,t) = \frac{d}{dt}\big(t \int_{S_t(0)} \varphi dS\big) + t \int_{S_t(0)} \psi ds$$

and now we have ourselves Kinchhoff formula

$$u(x,t) = \frac{d}{dt}(t \int_{S_t(x)} \varphi ds) + t \int_{S_t(x)} \psi ds$$

or can be expressed as

$$u(x,t) = \int_{S_r(0)} \varphi + t\varphi_\nu + t\psi dS$$

after we compute produce rule.

Continuing from last discussion, we further explore 3D wave equation. Consider

$$u_{tt} = \triangle u$$

$$u(x,0) = \varphi(x)$$

$$u_t(x,0) = \psi(x)$$

while $u(x,t)$ is defined $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ and $t\in[0,,\infty)$. Using kinchhoff's formula,

$$\begin{aligned} u(x,t) &= \frac{d}{dt}\big(t\int_{S_t(x)}\varphi ds\big)+t\int_{S_t(x)}\psi ds \\ &= \int_{S_t}\varphi+t\varphi_\nu+t\psi dS \end{aligned}$$

so $\varphi_\nu=\nabla\varphi\nu$ while ν outer normal to the sphere where $S_t(x)=\{y:|x-y|=t\}$. Need to assume that φ and ψ are supported in the unit ball, $B_1=\{x||x|\leq 1\}$ (φ and ψ are 0 outside B_1). Hence,

$$u(x,t)=\begin{cases} 0, & t<|x|-1 \\ \sim|x|^{-1}, & |x|-1\leq t\leq|x|+1 \\ 0, & t>|x|+1 \end{cases}$$

note that $u(x,t)\sim\frac{|x|}{|x|^2}$ since we note that $\max|\varphi|,|\psi|,|\nabla\varphi|\leq 1$.

However, if we do change of variables, $t\to ct$, $\varphi\to\varphi$ and $\psi\to\frac{1}{c}\psi$, we would have

$$u(x,t)=\frac{d}{dt}\big(t\int_{S_{ct}(x)}\varphi ds\big)+t\int_{S_{ct}(x)}\psi dS$$

with a similar equation.

Taking a step back, we discuss 2D wave equation

$$u(x_1,x_2,t),u_{tt}=\triangle u$$

with initial condition

$$u(x_1,x_2,0)=\varphi(x_1,x_2)$$
$$u_t(x_1,x_2,0)=\psi(x_1,x_2)$$

Extend u to a function in $\mathbb{R}^3$ to a function in $\mathbb{R}^3$ constant in the extra variable $v(x_1,x_2,x_3,t)=u(x_1,x_2,t)$. By Kinchoff's formula

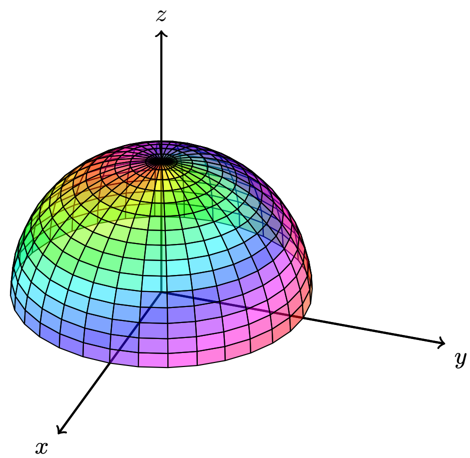
$$u(x_1,x_2,t)=\frac{d}{dt}\big(t\int_{S_t(x)}\varphi dS\big)+t\int_{S_t(x)}\psi dS$$

That is, imagine a 3D sphere and over it we have $t\int_{S_t(0)}\varphi(x_1,x_2)dS$ and we can write

$$t\int_{S_t(0)}\varphi(x_1,x_2)dS=\frac{t}{4\pi t^2}\int_{S_t(0)}\varphi(x_1,x_2)dS$$

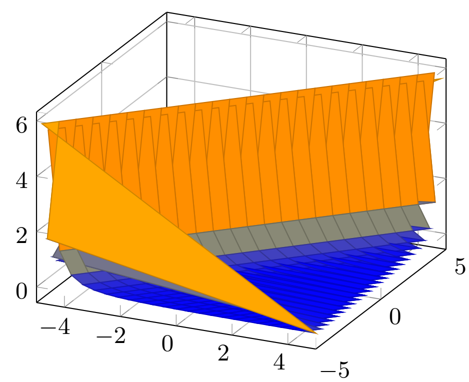
and we need to figure out an expression for dS using the Jacobian which is a function of (x_1,x_2,t). The normal vector ν is a component of (ν_1,ν_2) from 2D point of view. That is, we have

$$\begin{aligned} \tfrac{t}{4\pi t^2}\textstyle\int_{S_t(0)}\varphi(x_1,x_2)dS &= \frac{1}{4\pi}\int\varphi(x_1,x_2)\frac{t}{\sqrt{t^2-|x|^2}}dx_1dx_2 \\ &= \frac{d}{dt}\left(\frac{1}{2\pi}\int_{B_t(x)}\varphi(y_1,y_2)\frac{1}{\sqrt{t^2-|x-y|^2}}dy_1dy_2\right) \end{aligned}$$



To count points on a ball in 3D, we need to integrate data by computing how much do these points count. We imagine the following surface

$$\frac{1}{\sqrt{t^2 - |y-x|^2}} \sim$$



4 Heat Equation

4.1 Introduction

We consider $u_{xx} - u_y = 0$, which is usually written $u_t = u_{xx}$. This is the heat equation. It takes the following form $\int_T udx - \int_B udx = \int_R u_x dt - \int_L u_x dt$. Then

$$\begin{aligned} \int_T udx - \int_B udx &= \int_R u_x dt - \int_L u_x dt \\ \iint_Q u_t dxdt &= \iint_Q (u_x) x dxdt \end{aligned}$$

which becomes $u_t = ku_{xx}$ which is called heat equation or diffusion equation. In this case, the boundary condition, $u(x,t)$ with $x \in [0, L], t \in [0, \infty]$. Then $u_t = u_{xx}$ and $u(0,x) = \varphi(x)$ given. Now say there is $u(t,0) = q(t)$.

Let us discuss 1D wave equation. Consider the following

$$u(x,t), x \in [0,L], t \in [0,\infty)$$

$$u_t = u_{xx} \text{ in } (0,L) \times (0,\infty) \text{ with}$$

with boundary condition

$$u(x,0) = \varphi(x), u(0,t) = g(t), u(L,t) = h(t)$$

4.2 An Example

Consider another form of heat equation:

$$u_t = u_{xx}$$

in $(0,L) \times (0,\infty)$. Note the boundary condition

$$u(x,0) = \varphi(x), u(t,0) = g(t), u(t,L) = h(t)$$

The question is: whether we have uniqueness and stability. From here we can discuss some important properties such as maximum principal.

Proposition 4.2.1. ***Maximum Principal***. *The maximum (minimum) of $u(x,t)$ in $[0,L] \times [0,T]$ occurs achieved either at $t = 0$ (bottom) or at $x = 0$ or $x = L$ (on the sides).*

Proof. Let (x_0, t_0) be a point where the max of u is achieved and assumei ti s not on the bottom of the on the sides.

$$u_{xx}(x_0,t_0) \leq 0, u_t(x_0,t_0) \geq 0$$

almost a contradiction if we know that $u_t < u_{xx}$ then we would have a contradiction.

We want to work with $v(x,t) = u(x,t) - \epsilon t$ for some $\epsilon > 0$ small. That way we have

$$v_t = u_t - \epsilon, v_{xx} = u_{xx}$$

$$\Rightarrow v_t < v_{xx}$$

The first part holds for v

$$\underbrace{\max}_{[0,t]\times[0,L]} v \leq \underbrace{\max}_{\text{bottom 2 sides}} v$$

We let $\epsilon \to 0$ and obtain the same inequality for u.

Remark 4.2.2. We only used that $u_t \leq u_{xx}$.

□

4.3 Uniqueness

Consider alternatively say we want to prove uniqueness,

$$u^i_t = u^i_{xx}$$

$$u^i(x,0) = \varphi^i(x)$$
$$u^i(0,t) = g^i(x)$$
$$u^i(0,t) = g^i(x)$$
$$u^i(L,t) = h^i(x)$$

So we consider the difference $w = u' - u^2$ while $w_t = w_{xx}$. Then $w(x,0) = \varphi'(x) - \varphi^i(x)$, $w(0,t) = g'(t) - t^2(t)$, and $W(L,t) = h'(t) - h^2(t)$. For uniqueness and stability we can reduce to the case of

$$\max w = 0 \text{ and } \min w = 0$$

$$\Rightarrow w = 0$$

everywhere. The second proof of uniqueness and stability

$$u_t = u_{xx}$$

and we have

$$\begin{aligned} 0 &= \int (u_t - u_{xx}) u dx \\ &= \int (\frac{1}{2} u^3) - u_{xx} u dx \\ &= \frac{\partial}{\partial t}\left(\int \frac{1}{2} u^2 dx \right) + \int u_x u_x dx - u_x u|_{x=0}^{x-L} \end{aligned}$$

Integrating by parts

$$\int_)^L -u_{xx} dx = \int_0^L u_x^2 dx - u u_x|_{x=0}^{x=L}$$

and we can conclude assume $u = 0$ on the sides (or $u_x = 0$ on the sides

$$\frac{d}{dt} E(t) = -\int_0^L u_x^2 dx \leq 0$$

when $E(t) = \int_0^L \frac{1}{2} u^2(x,t) dx \geq 0$, thus $E(t)$ is decreasing in t. If the initial data $\varphi \equiv 0$ as well then $E(0) = 0$ and then $E(t) = 0$ for all $t \geq 0$, which means $u(x,t) = 0$ for all $t \geq 0$. If φ is not 0, then we have

$$\frac{1}{2}\int_0^L u^2(x,t) dx \leq \frac{1}{2}\int_0^L \varphi^2(x) dx$$

Consider the following example

$$u_t = u_{xx}, u(x,0) = \varphi(x)$$

with $x \in \mathbb{R}$ and $t \in [0, \infty)$. Say you start with initial data (imagine a bar on an axis). We want to solve first the heat equation with data φ concentrated at a point $\varphi = \delta(0)$ which is a function of integral. Then we have the following observations

(1) Total heat would be $\int_{-\infty}^{\infty} u(x,t) dx$ is preserved in time

(2) Fact: $u(x,t) \geq 0$ for all t if $\varphi \geq 0$ by maximum principle.

(3) For any $\lambda > 0$, $u(\lambda x, \lambda^2 t)$ solves the same equation.Then $\lambda u(\lambda x, \lambda^2 t)$ has initial data $\lambda\varphi(\lambda x)$ which has the same intended as $\varphi(x)$.

We conclude that for any λ, write the following (call it $\star$),

$$\lambda u(\lambda x, \lambda^2 t) = u(x,t)$$

if the initial data is $\delta(0)$. Denote by $g(x) = u(x,1)$ the profile of u at time 1. Put $\lambda = t^{-1/2}$ in $\star$. Then

$$u(x,t) = t^{-1/2} u(x t^{-1/2}, 1) = t^{-1/2} g(x t^{-1/2})$$

Then

$$u(x,t) = t^{-1/2} g(x t^{-1/2})$$

where g is a nonnegative function of integral. Find g such that u solves the heat equation

$$u_t = -\frac{1}{2}t^{-3/2}g(xt^{-1/2}) + t^{-1/2}g'(xt^{-1/2})(-\frac{1}{2})xt^{-3/2}$$

$$u_{xx} = t^{-3/2}g''(xt^{-1/2})$$

Let $s = xt^{-1/2}$, and

$$u_t = -\frac{1}{2}t^{3/2}(g(s) + g'(s)S)$$

$$u_{xx} = t^{-3/2}g''(s)$$

Next we reduce second order to first order

$$g'' = -\frac{1}{2}(sg)'$$

$$g' = -\frac{1}{2}sg + \text{constant}$$

and in this case the constant is 0 by symmetry (g is even). We are dealing with

$$g' = -\frac{1}{2}sg \Rightarrow (\ln g)' = -\frac{1}{2}s$$

$$\ln g = -\frac{1}{4}s^2 + \text{constant}$$

$$g(s) = c_0 e^{-\frac{1}{4}s^2}$$

and we have ourselves a general solution. We find c_0 from $\int_{-\infty}^{\infty} g(s)ds = 1$ and we want to solve

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4}s^2} ds = 2\sqrt{\pi}$$

and note the trick $\iint e^{-(x_1^2+x_2^2)}dx_1dx_2$ and we get $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$. Then we get

$$g(s) = \frac{1}{\sqrt{4\pi}}e^{-\frac{s^2}{4}}$$

and $u(x,t) = t^{-1/2}g(xt^{-1/2})$ gives us the heat kernel

$$s(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$$

or the fundamental solution of the heat equation.

Remark 4.3.1. We want to deduce heat equation, a solution of a PDE. The form of the situation is stretched by λ. Somehow when we do this the solution will be easily solved (we are killing time while not changing the solution).

4.4 Convolution

Now we discuss some properties of $S(x,t)$:

(1) $\int_{-\infty}^{\infty} S(x,t)dx = 1$

(2) $s(x,t) \geq 0$

(3) $S(\lambda x, \lambda^2 t) = S(x,t)$

Convolution between two functions f and g defined on $\mathbb{R}$ with $f : \mathbb{R} \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$. Now we have

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

With this notation the formula $\star$ can be written

$$u(x,t) = (\varphi \star S(\cdot,t))(x)$$

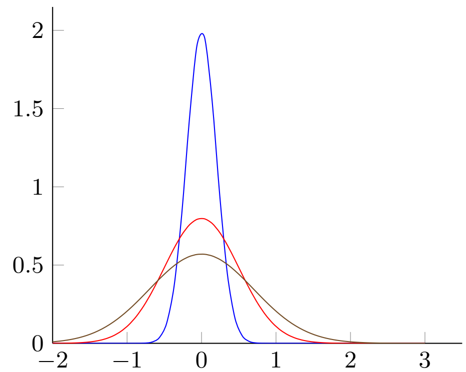
Consider standard form of heat equation

$$u_t = u_{xx}, u(x,0) = \varphi(x)$$

for $x \in \mathbb{R}$, and $t \geq 0$. We have solution

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \varphi(y)dy = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)dy = S(\cdot,t) \star \varphi$$

with the last step called convolution, which is referred to page 49 from text [1].



This is like we plot a family of normal distribution and each with a scaling property. We either stretch the graph out or press the graph down.

This will solve us

$$S(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2} 4t$$

and we need to check the integral formula which is stated in the remark below.

Remark 4.4.1. The integral makes sense if

$$|\varphi(y)| \leq M$$

for all y. More generally,

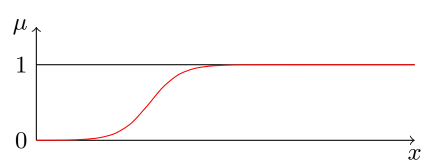
$$|\varphi(y)| \leq Me^{|y|}$$

or we can state

$$|\varphi(y)| \leq Me^{c|y|}$$

say you grow very fast exponentially, which is the case the data grow as fast as the function decays.

Remark 4.4.2. If $u(x,t)$ is infinitely differentiable for $t > 0$ even if the initial data φ is only continuous. One can check d'ALembert's formula. The jump in the center will get smoothed out by x.



Remark 4.4.3. Infinite speed of propagation. Heat equation has "infinite speed" of propagation. If $\varphi \geq 0$ and $\varphi = 0$ outside $[-1, 1]$, then $u(x,t) > 0$ for all x no matter how small t is.

Remark 4.4.4. There is no uniqueness. Consider

$$u_t = u_{xx}, u(x,0) = 0$$

would give us $u(x,t) = 0$ if we are given the sides (edges). Thus, there is no uniqueness in general.

Alternatively, consider

$$u_t = u_{xx}, u(x,0) = 0, |u(x,t)| \leq Me^{c|x|}$$

4.5 An Application: Random Walk

Random Walk. At step 0, you are at a point (say origin) with probability 1. The next time, step 1, you jump left or right with probability half. Continuing with such process, step 2 you jump to left or right with probability half again. At step m, we have probably at one node to be $\frac{1}{2^m}$. This procedure, in the middle node at step m, the probability will be $\binom{m}{k}2^{-m}$. We count the population with becomes a binomial distribution. It would form a histogram that follows a fitted normal distribution with summation of such area as 1. Then we have

$$u(x, t + J) = \frac{1}{2}u(x - h, t) + \frac{1}{2}u(x + h, t)$$

then we can claim that this is a heat equation. Then we do the following

$$u(x, t + J) - u(x,t) = \frac{1}{2}\Big(u(x - h, t) + u(x + h, t) - 2u(x,t)\Big)$$

and

$$Ju_t = h^2 u_{xx}$$

5 Schrodinger Equation

For $u : \mathbb{R}^2 \to \mathbb{C}$ a function of t and x, we have $-iu_t = u_{xx}$ which is Schrodinger equation.

6 Fourier Series

We discuss the procedures how Fourier series solve heat equations and wave equations.

Consider heat equation $u_t = u_{xx}$ in finite interval $x \in [0, l]$ and $t \in [0, \infty)$. Call this set up $\star$. Assume boundary conditions

$$u(x, 0) = \varphi(x), u(0,t) = u(l,t) = 0$$

and we have solution

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \varphi_{\varphi \times t}(y) dy$$

while the extension of φ to the whole real line $\mathbb{R}$. This will be a graph of data reflected oddly from the interval $[0, l]$.

Let us find some particular solutions of $\star$ of the form

$$u(x,t) = w(x)h(t)$$

which is a profile multiplied by $h(t)$, a function of time. The shape will preserved but intensity will be changed in time t. Let boundary condition be $w(0) = w(l) = 0$. Then

$$u_t = w(x)h'(t), u_{xx} = w''(x)h(t)$$

We want $w''h(t) = w(x)h'(t)$. That is, we have

$$\frac{w''(x)}{w(x)} = \frac{h'(t)}{h(t)} = -\lambda$$

which is a constant. We want to solve $w''(x) = -\lambda w$. This is a second-order ODE. Discuss the following.

If $\lambda < 0$, then $-\lambda > 0$. Then

$$w(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

Then $A = B = 0$. We can check this by looking at the boundary condition and also notice that $\lambda \neq 0$.

If $\lambda = 0$, then $w'' = 0$. Integrating twice we get $w(x) = A + Bx$. Then by boundary condition, $w(0) = w(l) = 0$, then we have $A = B = 0$. Thus $w \equiv 0$.

If $\lambda > 0$, then $-\lambda < 0$. Then we have

$$w(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

which by, $w(0) = 0$, implies $B = 0$. Note that $w(l) = 0$, then $A\sin(\sqrt{\lambda}l) = 0$. If $\sin(\sqrt{\lambda}l) = 0$, then w can be nonzero, $\sqrt{\lambda}l = n\pi$ for $n = 1, 2, 3, \ldots$.

Thus, we conclude if $\lambda = \lambda_n = \left(\frac{n\pi}{\lambda}\right)^2$, then

$$w(x) = A_n \sin(\frac{n\pi}{\lambda}x)$$

and then

$$\frac{h'(t)}{h(t)} = -\lambda_n \Rightarrow h(t) = ce^{-\lambda_n t}$$

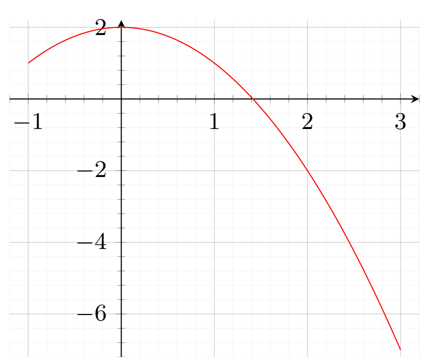
From boundary condition $w(0) = w(l) = 0$ note that $h(0) = 1$. Thus

$$h(t) = e^{-\left(\frac{n\pi}{l}\right)^2 t}$$

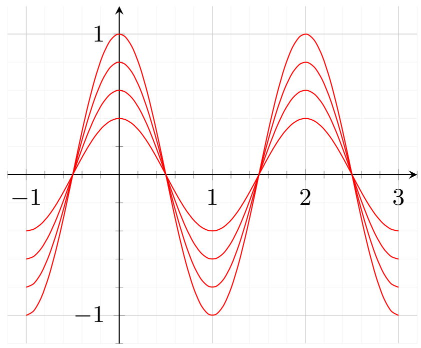
and

$$A_n \sin(\frac{n\pi}{l}x)e^{-\left(\frac{n\pi}{l}\right)^2 t}$$

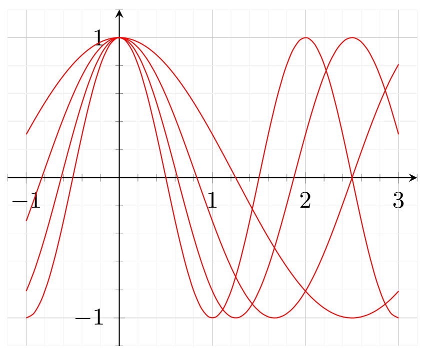
for $n = 1, 2, 3, \ldots$. We get, if $n = 1$, a quadratic curve, namely $w_1 = \sin(\frac{\pi}{l}x)$.



and moreover we have $w_2(x) = \sin(\frac{2\pi}{l}x)$ while $h_2(t) = e^{-\left(\frac{2\pi}{l}\right)^2 t}$ for $n = 2$.



and next $w_n(x) = \sin(\frac{n\pi}{l}x$ while $h_n(t) = e^{-\left(\frac{n\pi}{l}\right)^2 t}$, which will shrinks faster and have more bumps within the interval $[0, l]$.



If $\varphi = w_1 + 3w_5 - 7w_9$, then the solution is

$$w(x,t) = w_1 h_1(t) + 3w_5(t)h_t(t) + 7w_9(x)h_9(t)$$

which will be

$$u(x,t) = w_1 e^{-(\frac{\pi}{l})^2 t} + 3w_5(x)e^{-(\frac{5\pi}{l})^2 t} - 7w_9(x)e^{-(\frac{9\pi}{l})^2 t}$$

We ask the following question. Can we write a function $\varphi(x)$ with $\varphi(0) = \varphi(l) = 0$, a sum of w_n's. Then do we have the following

$$\varphi(x) = \sum_{n=1}^{\infty} A_n w_n$$

which looks simple if you think of functions of superposition of n.

6.1 Newmann Boundary

Consider Newmann boundary conditions

$$u_t = u_{xx}, u(x,0) = \varphi(x), u_x(0,t) = u_x(l,t) = 0$$

and you can check case by case just be the procedures above. We discuss

Case 1, assume $\lambda < 0$, which gives us nothing, i.e. $w = 0$.

Case 2, assume $\lambda = 0$, which gives us $w(x) = A + Bx$

Case 3, assume $\lambda > 0$, which gives us $w(x) = A\sin(\frac{n\pi}{l}x) + B\cos(\frac{n\pi}{l}x)$. Then $w'(0) = w'(l) = 0$ which implies that $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ for $n = 1, 2, 3, \ldots$. Next, $\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2$ which implies that $w_n(x) = \cos\left(\frac{n\pi}{l}x\right)$ and then $h_n(t) = e^{-\lambda_n t}$.

Next question we could ask: if $\varphi'(0) = \varphi'(l) = 0$, can we write

$$\varphi(x) = \sum_{0}^{\infty} A_n w_n$$

while $w_n = \cos\left(\frac{n\pi}{l}x\right)$.

Consider $u_t = u_{xx}$ with boundary condition $u(x, 0) = \varphi(x)$. Then we consider the following.

(1) Dirichlet B.C.: $u(0, t) = u(l, t) = 0$;

(2) Newmann B.C.: $u_x(0, t) = u_x(l, t) = 0$.

Special solutions take the following form

$$u(x, t) = w(x)h(t)$$

which lead us to consider the second derivative problem,

$$-w'' = \lambda w$$

which becomes an eigenvalue problem for $\frac{d^2}{dx^2}$ while $w \neq 0$, w eigen function and λ eigen value. Then we consider

$$\lambda_1 = \left(\frac{\pi}{l}\right)^2, \lambda_2 = \left(\frac{2\pi}{l}\right)^2, \ldots, \lambda_n = \left(\frac{n\pi}{l}\right)^2, \text{ while } w_n = \sin(\left(\frac{n\pi}{l}x\right)$$

Next, we have

$$\lambda_0 = 0, w_0 = 1$$
$$\lambda_n = \left(\frac{\pi}{l}\right)^2, w_n = \cos\left(\frac{n\pi}{l}x\right)$$

while $h_n(t) = e^{-\lambda_n t}$.

6.2 Robin Conditions

Thirdly, we discuss Robin conditions and we have

$$u_x(0, t) - a_0 u(0, t) = 0$$

$$u_x(l, t) + a_l u(0, t) = 0$$

while a_0, a_l are constants. We get different values for λ_n's. Refer to Section 4.3 (page 92) of the text [1].

We can also have functions to be periodic.

$$u(0, t) = u(l, t)$$

$$u_x(0, t) = u_x(l, t)$$

then we compute

$$\lambda_0 = 0, w_0 = 1$$
$$\lambda_1 = \lambda_2 = \left(\frac{2\pi}{l}\right)^2, w_1 = \sin\frac{2\pi}{l}x, w_2 = \cos\left(\frac{2\pi}{l}x\right)$$
$$\lambda_3 = \lambda_4 = \left(2\frac{2\pi}{l}\right)^2, w_3 = \sin\left(2\frac{2\pi}{l}x\right)^2, w_4 = \cos\left(2\frac{2\pi}{l}x\right)^2$$

and we continue until

$$\lambda_{2n-1} = \lambda_{2n} = \left(n\frac{2\pi}{l}\right)^2, w_{2n-1} = \sin\left(n\frac{2\pi}{l}x\right), w_{2n} = \cos\left(n\frac{2\pi}{l}x\right)$$

Remark 6.2.1. In page 84 text [1], X stands for function w, and T stands for $h(t)$.

6.3 Fourier in Wave Equation

How about wave equations? Consider wave equation $u_{tt} = u_{xx}$ for $x \in [0, l]$, $t \in [0, \infty)$. Also given conditions $u(x, 0) = \varphi(x)$ and $u_t(x, 0) = \psi(x)$ with I, II, III, and IV boundary conditions. Next,

$$w(x)h''(t) = w''(x)h(t)$$

Then we have

$$\frac{w''(x)}{w(x)} = \frac{h''(t)}{h(t)} = -\lambda$$

while λ is a constant. If $\lambda = \lambda_n$ is an eigenvalue, then

$$\begin{aligned} h_n(t) &= A_n \cos(\sqrt{\lambda_n} t) + B_n \sin(\sqrt{\lambda_n} t), \lambda_n > 0 \\ h_n(t) &= A_n + B_n t, \lambda_n = 0 \\ h_n(t) &= A_n e^{\sqrt{-\lambda} t} + B_n e^{-\sqrt{-\lambda} t}, \lambda_n < 0 \end{aligned}$$

If we can write

$$\varphi(x) = \sum C_n \sin\left(\sqrt{\lambda_n} x\right)$$

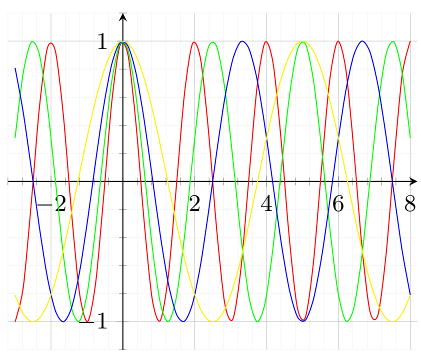
$$\psi(x) = \sum D_n \sin\left(\sqrt{\lambda_n} x\right)$$

and we solve

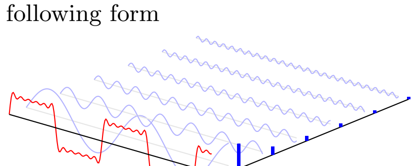
$$\sum (A_n \cos(\sqrt{\lambda_n} t) + B_n \sin(\sqrt{\lambda_n} t)) \sin(\sqrt{\lambda_n} x)$$

We find A_n and B_n from plugging $t = 0$. We solve for $A_n = C_n$ and $B_n = D_n$.

For notation purpose, let us write $l = \pi$. Assume f is a periodic function of period 2π.



or one can consider the following form



Then we can compute eigenvalues of $-w'' = \lambda w$ and

$$0 \to 1$$

$$1 \to \sin x \text{ and } \cos x$$
$$2^2 \to \sin 2x \text{ and } \cos 2x$$

and so on. The question is can we write (approximate a trigonometric polynomial)

$$f(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$$

Definition 6.3.1. If f and g are periodic of period 2π, we define

$$f \cdot g \stackrel{\text{def}}{=} \int_{-pi}^{\pi} f(x)g(x)dx$$

We have a key property

Proposition 6.3.2. *We have $w_n \cdot w_m = 0$ if $n \neq m$ and*

$$w_n \in \{1, \cos x, \sin x, \cos 2x, \sin 2x, ...\}$$

$$\sin nx \sin mx = \frac{1}{2m} = \frac{1}{2}[\cos(n+m)x - \cos(n+m)]$$

(eigenfunctions are orthogonal to each other

If $\star$ were time, then we integrate

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{1}{2}A_0 = \pi A_0$$

and solve for

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi} f \cdot 1$$

considering dot product to be the approach of projecting function to a direction. In this case,

$$\begin{aligned} f \cdot w_n &= f \cdot \cos(nx) = \int_{-\pi}^{\pi} A_n[\cos(nx)]^2 dx \\ A_n &= \tfrac{1}{\pi} f \cdot \cos(nx) \end{aligned}$$

which solves for A_n.

Consider $f : \mathbb{R} \to \mathbb{R}$ periodic of period 2π. Also $f = A_0\frac{1}{2} + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$ for some constants A_0, A_n, B_n for $n \geq 1$. Consider $\frac{1}{2}$, $\cos x, \sin x$, $\cos 2x$, $\sin 2x$, and so on are eigen functions for the ODE. Then

$$-w'' = \lambda w$$

$$f \cdot g = \int_{-\pi}^{\pi} f(x)g(x)dx$$

Any two functions in the sequence above are orthogonal to each other (dot product is zero). If A is symmetric matrix, then eigen vectors are orthogonal, i.e. $AV = \lambda V$.

Fourier coefficients

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi} f \cdot (\cos nx)$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi} f \cdot (\sin nx)$$

Example 6.3.3. Consider $f(x) = x$ in $[-\pi,)$ extended periodically. Then we have

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)dx = 0$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)dx = 0$$

and solve

$$\begin{aligned} B_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} x\left(\tfrac{-\cos nx}{n}\right)' dx \\ &= \tfrac{1}{\pi} - \tfrac{1}{n}\cos nx\Big|_{-\pi}^{\pi} - \tfrac{1}{\pi}\int_{-\pi}^{\pi} -\tfrac{1}{n}\cos nx dx \\ &= \tfrac{2\pi(-1)^n}{n\pi} = \tfrac{2}{n}(-1)^{n+1} \end{aligned}$$

Fourier series of x is

$$\sum_{1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)$$
$$= 2(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots)$$

Complex notation.

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos x - i \sin x$$

and $f, g : \mathbb{R} \to \mathbb{C}$ then

$$f \cdot g = \int_a^b f(x)g(x)dx$$

We can use as eigen function

$$\underbrace{1}_{0}, \underbrace{e^{ix}, e^{-itx}}_{1}, \underbrace{e^{i2x}, e^{-i2x}}_{2^2}, \dots$$

and we have e^{inx} for $n \in \mathbb{Z}$. Then we have Fourier series written in complex

$$f = \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

while

$$a_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

Denote by

$$S_N(f) = \frac{1}{2}A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx)$$

with A_n, B_n the Fourier coefficients of f. We prove convergence theorem which is in page 126 [1].

Theorem 6.3.4. *L^2 convergence in mean square)*

$$||f - S_N(f)||^2 = \int_{-\pi}^{\pi} |f - S_N(f)|^2 dx \to 0 \text{ as } N \to \infty$$

while $||f|| = \sqrt{f \cdot f} = \left(\int_{-\pi}^{\pi} f^2 dx\right)^{1/2}$ *and to approximate, we do*

$$\sum_{n=1}^{N} \frac{2}{n}(-1)^{n+1}\sin(nx)$$

Theorem 6.3.5. *Uniform convergence: There are two parts:*
If $f \in C1$ and periodic, then

$$\max_{[-\pi,\pi]} |f - S_N(f)| \to 0 \text{ as } N \to \infty$$

uniform convergence.

Theorem 6.3.6. *Point-wise convergence:*
If f is piece wise C^1 (with possible discontinuities) near a point x, then

$$S_N(f)(x) \to \frac{1}{2}[f(x^+) + f(x^-)]$$

as $N \to \infty$ where $f(x^+)$ and $f(x^-)$ are the limits of f at x from right and left.

Proof. Let X be a vector space with inner product and let $w_1, w_2, ..., w_N$ be orthogonal vectors (nonzero).

$$w_i \cdot w_j = 0 \text{ if } i \neq j$$

Then the minimum of

$$||w - \sum_{i=1}^{N} a_i w_i||$$

is realized for the choice $a_i = \frac{w \cdot w_i}{w_i \cdot w_i} = \frac{w \cdot w_i}{||w_i||^2}$ which is the w_i's Fourier coefficient of w. Also $||w||^2 = ||w - \sum a_i w_i||^2 + \sum_{i=1}^{N} a_i^2 ||w_i||^2$ □

Proof. Denote $\sum -i = 1^N a_i w_i = v$ with a_i as defined above. Then

$$(w - v) \cdot w_i = 0 \text{ for } i = 1, 2, ..., n$$

and $v w_i = a_i (w_i w_i)$.

Let $y = \sum_{i=1}^{n} b_i w_i$ we have $(W - v) \cdot y = 0$ and then

$$w - y = w - v + v - y$$

$$\Rightarrow (w - y) \cdot (w - y) = (w - v) \cdot (w - v) + (v - y) \cdot (v - y) + 2(w - v) \cdot (v - y)$$

and we have

$$||w - y||^2 = ||w - v||^2 + ||v - y||^2$$

□

6.4 Riemann-Lebesgue-Fourier

Riemann-Lebesgue-Fourier. Consider X a vector and we have $X \times X \to \mathbb{R}$ while $(f, g) \to f \cdot g \in \mathbb{R}$ which we want it to be linear associative such that this map is linear in each component

(1) satisfies

$$(f_1 + f_2) \cdot g = f_1 \cdot g + f_2 \cdot g$$

and more over

$$(\alpha_1 f_1 + \alpha_2 f_2) \cdot g = \alpha_1 f_1 \cdot g + \alpha_2 f_2 \cdot g$$

(2) dot product is greater or equal to zero

$$f \cdot f \geq 0$$

with equality only if $f = 0$, then

$$||f|| = \sqrt{f \cdot f}$$

defines a norm on X.

Proposition 6.4.1. *If $f \cdot g = 0$, then*

$$||f + g||^2 = ||f||^2 + ||g||^2$$

Proof. Start with

$$\begin{aligned} ||f + g||^2 &= ||f + g|| \cdot ||f + g|| \\ &= ff + fg + gf + gg \\ &= ||f||^2 + ||g||^2, \text{ while condition says } f \cdot g = 0 \end{aligned}$$

□

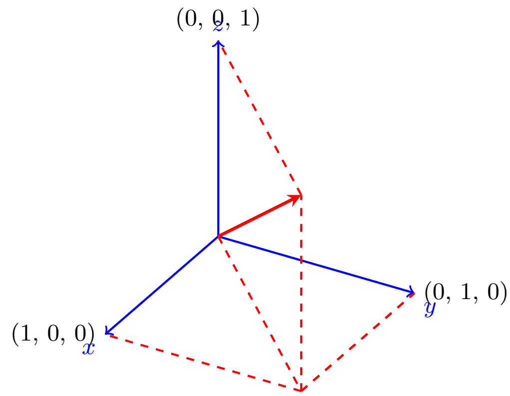
Consider X a space of continuous functions on $[-\pi, \pi]$ or $[a, b]$. Then

$$f \cdot g = \int_a^b f(x)g(x)dx$$

Theorem 6.4.2. *Consider X a vector space with inner product $\cdot$ while $w_1, ..., w_n$ orthogonal with each other (dot products are zero) and non-zero vectors*

$$||w - \sum_{i=1}^{n} a_i w_i||$$

Please refer to the following graph



for $a_i = \frac{w \cdot w_i}{w_i \cdot w_i}$. Moreover,

$$||w||^2 = ||w - \sum a_i w_i||^2 + \sum_{i=1}^{n} a_i^2 ||w_i||^2$$

which we can break down to the following formula

$$w = w - v + a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$$

Recall last time $v = \sum a_i w_i$ and $z = \sum b_i w_i$ for some other constants b_i. Then

$$(w - v) \cdot w_i = 0$$

which gives us

$$(w - v) \cdot (z - v) = 0 \Rightarrow ||w - z||^2 = ||w - v||^2 + ||v - z||^2$$

thus $||w - v||^2 \geq ||w - v||^2$ and $w = \sum a_i w_i$, while $ww_i = a_i(w_i \cdot w_i)$.

We solve

$$||f||^2 = ||f - S_N(f)||^2 + \pi \sum_{n=1}^{N} (A_n^2 + B_n^2)$$

$$\int_{-\pi}^{\pi} f^2 dx = \int_{-\pi}^{\pi} (f - S_N(f))^2 dx + \pi \sum_{n=1}^{N} (A_n^2 + B_n^2)$$

and we let $N \to \infty$,

$$\int_{-\pi}^{\pi} = \underbrace{\lim_{N\to\infty} \int_{-\pi}^{\pi} (f - S_N(f))^2 dx}_{\text{approximate to zero}} + \pi \sum_{i}^{\infty} A_n^2 + B_n^2$$

so we solve

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx \geq \sum_{1}^{n} A_n^2 + B_n^2$$

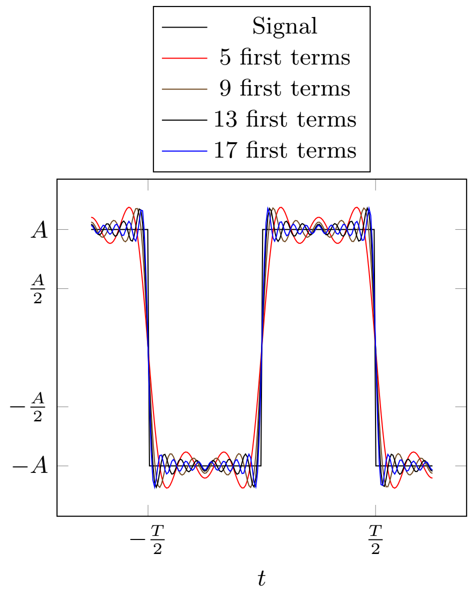
thus $A_n, B_n \to 0$ as $n \to \infty$.

We can conclude (Riemann-Lebesgue lemma)

$$\int_{-\pi}^{\pi} f(x)\sin(nx)dx \to 0$$

as $n \to \infty$. We write sine as $(-\frac{\cos(nx)}{n})^{-1}$ to prove it.

It is worth note that Riemann and Lebesgue work post significance in approximation, however, it is also noticeable that Fourier series can provide approximations to any functions as well. A typical graph illustration for Fourier series is below.



which is completeness from text [1]. Now let us prove this concept.

Proof. Let us prove point-wise convergence. If $f \in C'$ and periodic, then

$$S_N f(x) \to f(x) \text{ as } N \to \infty$$

Consider

$$S_N(f)(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx)$$

while

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)dy \leftarrow \cos(nx)$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)dy \leftarrow \sin(nx)$$

□

Plug A_n and B_n in the formula:

$$S_N(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\Big(1 + 2\cos(ny)\cos(nx) + 2\sin(ny)\cos(ny)\Big)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\underbrace{\left(1 + \sum_{n=1}^{N} 2\cos(n(y-x))\right)}_{\text{Dirichlet: } D_N(y-x)} dy$$

which becomes a formula with convolution

$$S_N(f)(x) = \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi} f(y)D_N(y-x)dx}_{f\star D_N \text{ a convolution}}$$

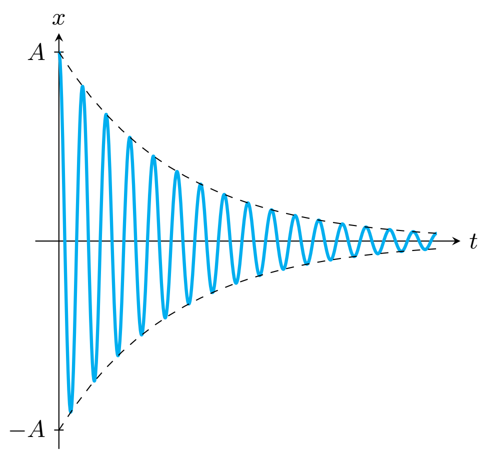
notice that $D_N(t)$ is the Dirichlet kernel which is

$$D_N(t) = 1 + 2\cos t + 2\cos(2t) + \cdots + 2\cos(Nt)$$

which can be written

$$D_N(t) = \frac{\sin[(N+\frac{1}{2})t]}{\sin(\frac{1}{2}t)}$$

with numerator oscillates faster than denominator which is page 137 from tex [1].



An important property is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)dt = 1$$

and then

$$S_N(f)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(y)-f(x))D_N(y-x)dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(y)-f(x))\frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t}dt$$

for $t = y - x$. Then we have

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(x+t)-f(x)}{\sin\frac{t}{2}}\sin(N+\frac{1}{2})tdt$$

by using Riemann-Lebesgue lemma.

Consider f periodic of period 2π while

$$f(x) \to \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx)$$

with coefficients

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \text{ for } n = 0, 1, 2, ...$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx, \text{ for } n = 1, 2, ...$$

while

$$S_N(f)(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N} A_n \cos nx + B_n \sin nx$$

Remark 6.4.3. Recall Theorems:

(1) Point-wise Convergence: Note that

$$S_N(f)(x) \to \frac{1}{2}\Big(f(x+) + f(x-)\Big)$$

if f is piece-wise C^1 mean x. There can be discontinuity. It is just required that at each point there is convergence even if it is at the discontinuity.

(2) Uniform Convergence: Note that

$$\max_x |S_N(f)(x) - f(x)| \to 0$$

if f is C^1. This means that for x one can always choose an interval such that the function converges to $f(x)$ absolutely.

(3) Mean Square Convergence: Note

$$\int_{-\pi}^{\pi} |S_N(f) - f|^2 dx \to 0$$

Proof. Let us prove. Consider

$$S_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)D_N(y)dy$$

$$D_N(y) = \frac{\sin(N+\frac{1}{2})y}{\sin\frac{1}{2}y}$$

How far is it away from $f(x)$? Then we consider

$$S_N(f)(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \big(f(x-y) - f(x)\big)\frac{\sin(N+\frac{1}{2})y}{\sin\frac{1}{2}y}dy$$

and we are interested to find out whether the final result goes to zero for this equation.

First, notice that if $f(x)$ exists then $S_N(f)(x) - f(x) \to 0$;
Then, if f is discontinuous at x but $f'(x+), f'(x-)$ exists, then$S_N(f)(x) - \frac{1}{2}\big(f(x+) + f(x-)\big) \to 0$. Just as Pythagoras theorem,

$$\int_{-\pi}^{\pi} f^2 dx = \int_{-\pi}^{\pi} |f - S_N(f)|^2 dx + \frac{\pi}{2}A_0^2 + \pi\sum_{1}^{N} A_n^2 + B_n^2$$

For $\epsilon > 0$ small, we find $\tilde{f} \in C^1$ such that

$$||f - \tilde{f}|| = \int_{-\pi}^{\pi} |f - \tilde{f}|^2 dx \leq \epsilon$$

$$\Rightarrow ||\tilde{f} - S_N(\tilde{f})|| \leq \epsilon$$

for large N by the uniform convergence. □

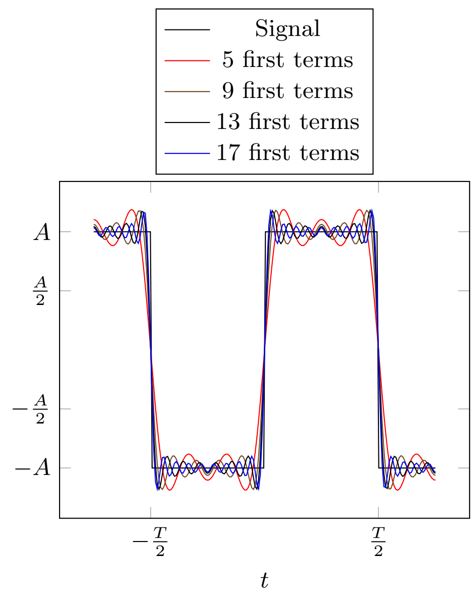
Parseval's theorem:

$$\int_{-\pi}^{\pi} f^2 dx = \frac{\pi}{2} A_0^2 + \pi \sum_1^{\infty} (A_n^2 + B_n^2)$$

Also recall last time: $f(x) = x \to A_n = 0, B_n = \frac{2}{n}(-1)^{n+1}$. How do we use Parseval's theorem?

$$\begin{aligned} \textstyle\int_{-\pi}^{\pi} x^2 dx &= \pi \sum_1^{\infty} (\frac{2}{n})^2 \\ &= \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \end{aligned}$$

Last thing about Fourier series: Given jump function



Consider $u_t = u_{xx}$ in $[0, \pi] \times [0, \infty)$. Also we have

$$u(0, t) = \cos t, u(\pi, t) = t, \text{ and } u(x, 0) = \sin x$$

while $\psi(x, t) = (1 - \frac{x}{\pi}) \sin t + \frac{x}{\pi} t$ then

$$\tilde{u}(x, t) = \psi(x_0, t) - \psi(x, t)$$

which gives

$$\tilde{u}(0, t) = 0 = \tilde{u}(\pi, t), \tilde{\varphi}(x, 0) = \sin x$$

Then consider

$$\tilde{u}_t = u_t - \psi_t, \tilde{u}_{xx} = u_{xx} - \psi_{xx}$$

$$\tilde{u}_t - \tilde{u}_{xx} = -(\psi_t - \psi_{xx}) = \tilde{f}(x,t)$$

while

$$\tilde{u}(x,t) = \sum_n B_n(t)\sin(nx), \tilde{f}(x,t) = \sum_n F_u(t)\sin(nx)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(B_n^1(t) - n^2 B_n(t)\right)\sin(nx) = \sum F_n(t)\sin(nx)$$

We want to solve B_n by ODEs,

$$B_n' - n^2 B_n = F_n$$

while $B_u(0)$ is given by the Fourier coefficients of $\tilde{\varphi}$. Solve by integrating factor

$$e^{-n^2 t}(B' - n^2 B) = Fe^{-n^2 t}$$

$$(e^{-n^2 t}B)' = e^{-n^2 t}F$$

7 Harmonic Functions

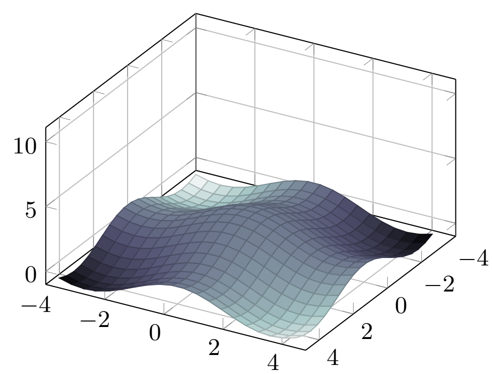
7.1 Harmonic Functions: Laplace Equation

Consider $u_{xx} = 0$ and $u'' = 0 \Rightarrow u' =$ const. Then we observe a one-dimensional graphical for $u(x)$. Yet the Laplace equation in two-dimensional space would be

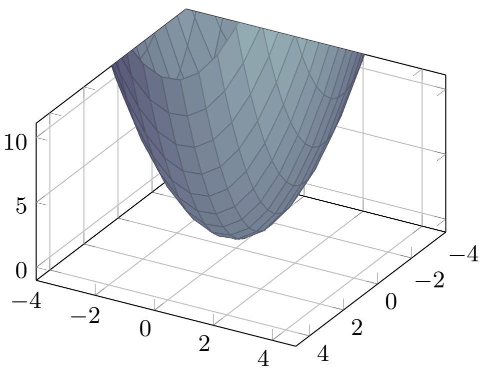
$$\triangle u = u_{xx} + u_{yy} = 0$$

The solutions to which are called harmonic functions. Then we have

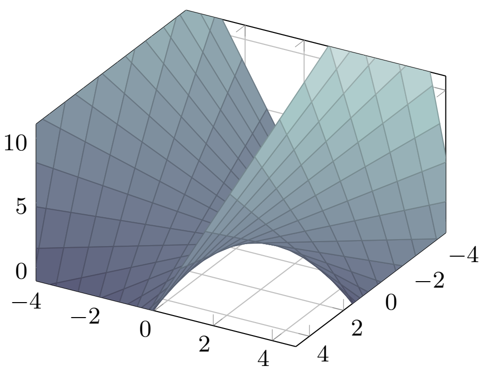
$$\begin{aligned} \triangle u &= \operatorname{div}(\nabla u) \\ &= (\partial_x, \partial_y)\cdot(u_x, u_y) \end{aligned}$$



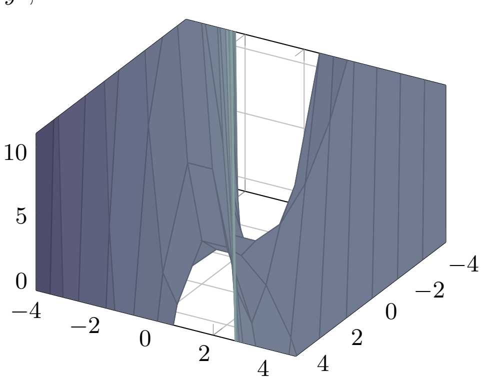
Example 7.1.1. Consider:
(1) $u(x,y) = ax + by + c$, a linear function;
(2) $u(x,y) = x^2 - y^2$;



(3) $u(x,y) = xy$;



(4) $u(x,y) = x^3 - 3xy^2$;



(5) $u(x,y) = h(x)g(y)$ and compute

$$\begin{aligned} u_{xx} + u_{yy} &= h''(x)g(y) + h(x)g''(y) \\ 0 &= h(x)g(y)\left(\frac{h''(x)}{h(x)} + \frac{g''(y)}{g(y)}\right) \end{aligned}$$

hence, we have

$$h(x)v = \sin(\lambda x) \text{ or } \cos(\lambda x)$$

$$g(y) = \exp(\lambda y) \text{ or } \exp(-\lambda y)$$

$$\sin(\lambda x)e^{\lambda y}, \sin(\lambda x)e^{-\lambda y}, \cos(\lambda x)e^{\lambda x}, \text{ or } \cos(\lambda x)e^{-\lambda y}$$

for $\lambda > 0$.

(6) This case it is related to complex numbers. Consider $f : \mathbb{C} \to \mathbb{C}$. That is,

$$z = x + iy \to f(z) = u + iv$$

$$(x, y) \to (u, v)$$
$$z \to f$$

Then $f'(z_0)$ means that $f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|)$. The Jacobi would be

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

while Df needs to be rotational-dilation matrix. That is, we have solution

$$= \begin{cases} u_x &= v_y \\ u_y &= -v_x \end{cases}$$

which is known as Cauchy-Riemann equation.

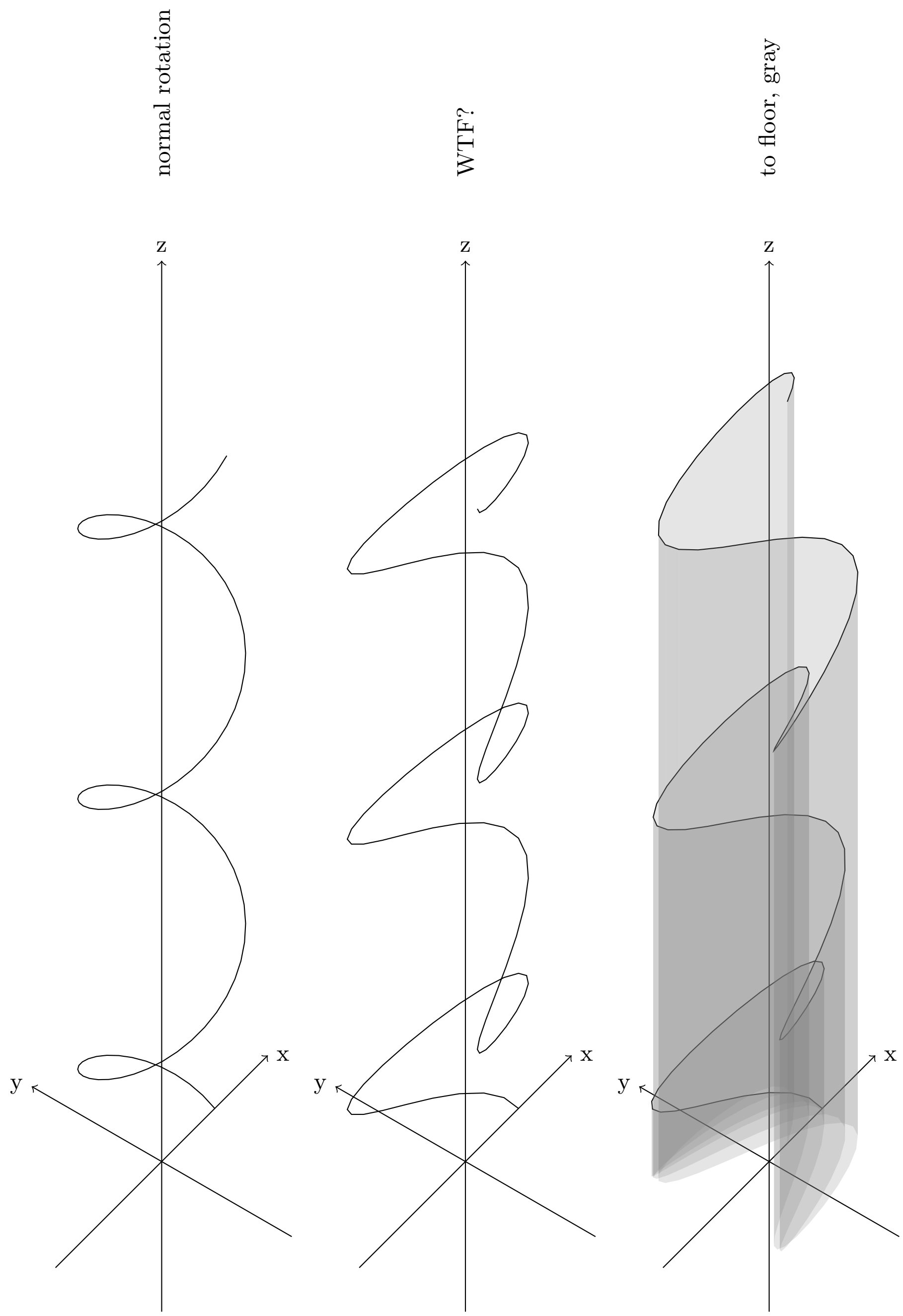
Then we from

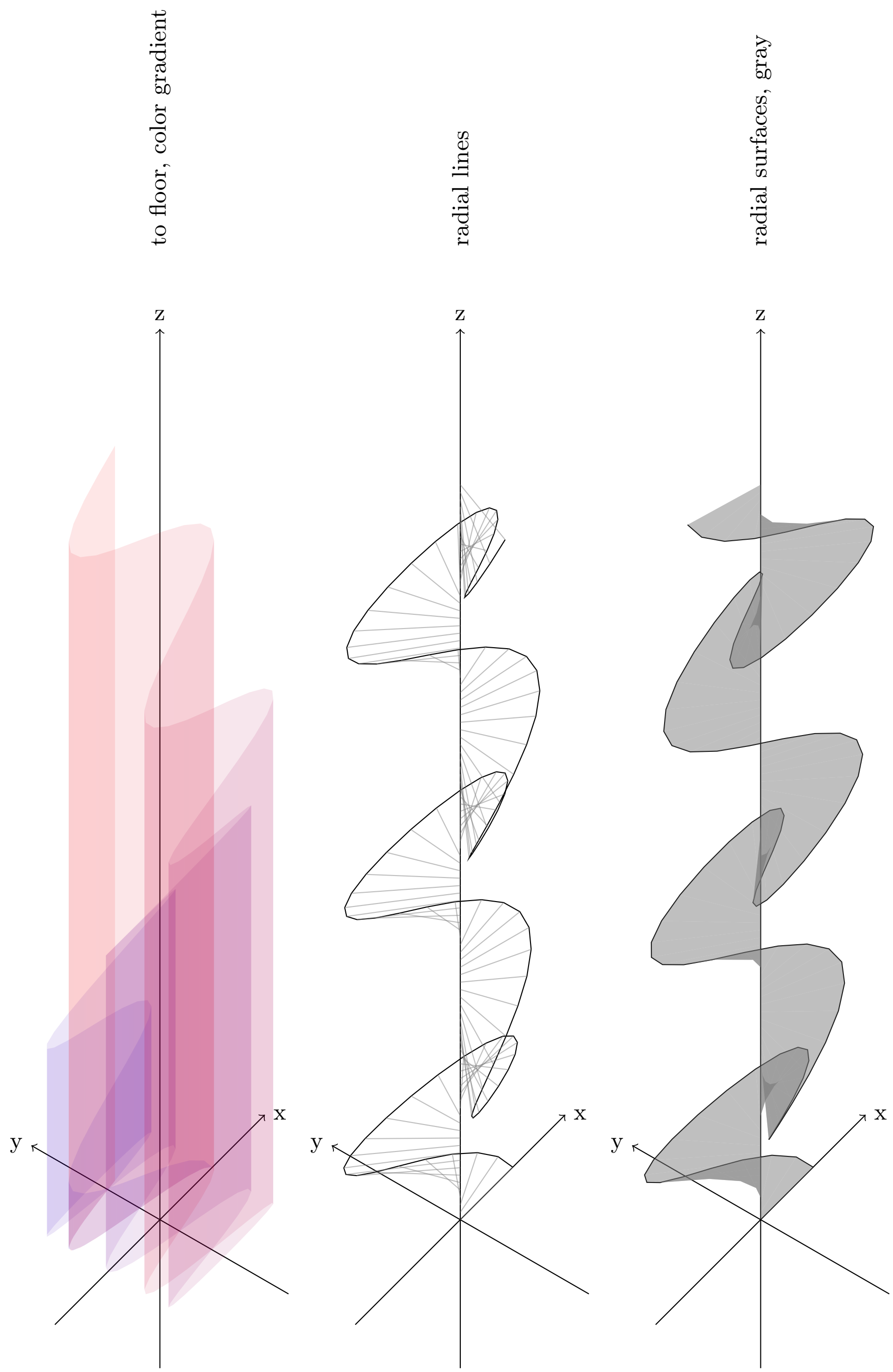
$$= \begin{cases} u_{xx} &= v_{yx} \\ u_{yy} &= -v_{xy} \end{cases}$$

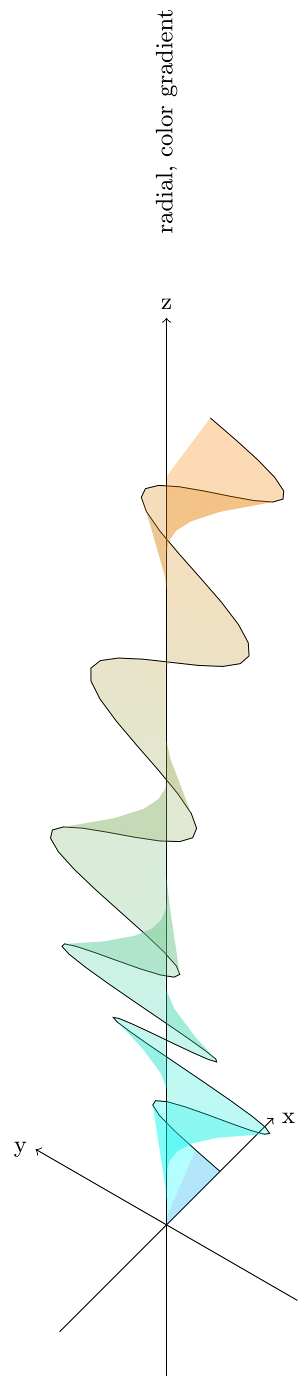
which gives us

$$= \begin{cases} u_{xx} + u_{yy} &= 0 \\ v_{xx} + v_{yy} &= 0 \end{cases}$$

If f is holomorphic ($f'(z)$ exists), then $u = \text{Re}f$ and $v = \text{Im}(f)$ are harmonic functions. Then $f(z) = z^n$ and $\text{Re}(z^n) = r^n \cos(n\theta)$ while (n, θ) are polar coordinates while imaginary part is $\text{Im}(z^n) = r^n \sin(n\theta)$.


normal rotation
WTF?
to floor, gray
z
z
z
x
y
x
y
x
y


to floor, color gradient
radial lines
radial surfaces, gray
z
x
y



7.2 Solve Dirichlet Problem

Let us recall Dirichlet problem: consider

$$\triangle u = 0 \text{ in } \Omega$$

$$u = \varphi \text{ on } \partial\Omega$$

we have a standard Dirichlet problem and we can find solution that satisfies the boundary condition φ.

Next, we can also go to Neumann problem such that

$$\triangle u = 0 \text{ in } \Omega, u_\nu = \psi \text{ on } \partial\Omega$$

We can also recall maximum principle. Assume u is continuous up to $\partial\omega$ and $\triangle u = 0$ in Ω. Then

$$\max_{\Omega} u = \max_{\partial\Omega} u$$
$$\min_{\Omega} u = \max_{\partial\Omega} u$$

and we can sketch a proof. At an interior maximum point, we have

$$u_{xx} \leq 0, u_{yy} \leq 0, \text{ and } u_{xx} + u_{yy} \leq 0$$

but consider special case that $u_{xx} = u_{yy} = 0$, then we would go to the saddle shape discussed above.

There is a trick: we work with $\tilde{u}(x, y) = u(x, y) + \epsilon x^2$ instead while $\epsilon > 0$ for some arbitrary small value. Then

$$\tilde{u}_{xx} = u_{xx} + 2\epsilon$$
$$\tilde{u}_{yy} = u_{yy}$$

and together we sum them up

$$\tilde{u}_{xx} + \tilde{u}_{yy} = u_{xx} + u_{yy} + 2\epsilon = 2\epsilon > 0$$

which is almost a contradiction. We can argue that this cannot have interior maximum. We can conclude that

$$\max_{\Omega} \tilde{u} = \max_{\partial\Omega} \tilde{u}$$

and by letting $\epsilon \to 0$ we have

$$\max_{\Omega} u = \max_{\partial\Omega} u$$

. Note that we would change the signs from plus to minus if we are working with the minimum.

A consequence following is the following:

(1) If

$$\triangle u = 0 \text{ on } \Omega, u = 0 \text{ on } \partial\Omega$$

then $u = 0$ in ω.

(2) Uniqueness for the Dirichlet problem:

$$\triangle u_1 = f \text{ in } \Omega, u_1 = \varphi \text{ on } \partial\Omega$$

$$\triangle u_2 = f \text{ in } \omega, u_2 = \varphi \text{ on } \partial\Omega$$

Then $u_1 - u_2 = 0$ satisfies $u_1 = u_2$.

For uniqueness by "energy" estimate, $\triangle u = 0$ in Ω, we have

$$\begin{aligned} 0 &= \int_\Omega (\triangle u) u dA \\ &= \int_\Omega u_{xx} u + u_{yy} u dA \\ &= -\int_\Omega u_x^2 + u_y^2 dA + \int_{\partial\Omega} u u_x \nu_x + u u_y \nu_y dA, \\ &\quad \text{where } \nu = (\nu_x, \nu_y) \text{ outer normal} \end{aligned}$$

Recall uniqueness of solutions to the Dirichlet problems. That is,

$$\triangle u = f \text{ in } \Omega, u = \varphi \text{ on } \partial\Omega$$

which is equivalent to show $u = 0$ is the only solution to Laplace $u = 0$ is the only solution to

$$\triangle u = 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega$$

Compute

$$\begin{aligned} 0 &= \int_\Omega (\triangle u) u dA \\ &= \int_\Omega (u_{xx} + u_{yy}) u dA \\ &= \int_\Omega u_{xx} u + u_{yy} u dA \\ &= -\int_\Omega (u_x u_x + u_y u_y) dA + \int_{\partial\Omega} u_x u \nu_x + u_y u \nu_y dS \end{aligned}$$

while $\nu = (\nu_x, \nu_y)$ outer unit normal to $\partial\Omega$, the boundary of Ω. Then

$$\int_\Omega (\triangle u) u dA = -\int_\Omega |\nabla u|^2 dA + \int_{\partial\Omega} u u_\nu dS$$

and we can conclude that

$$\int_\Omega |\nabla u|^2 dA = 0$$

which leads to $|\nabla u| = 0$ in Ω. This implies that

$$|\nabla u| = 0 \Rightarrow u_x = 0 \Rightarrow u \text{ is a constant in } x, y,$$

$$u_y = 0 \Rightarrow u \text{ a constant}$$

and $u = 0$ on $\partial\Omega$ implies $u = 0$ in Ω.

For Neumann problem,

$$\triangle u = 0, u_\nu = 0$$

Use this argument gives u is a constant function. Integrating by parts gives us, write it as (1),

$$\int_\Omega u_x dA = \int_{\partial\Omega} u \nu_x ds$$

$$\int_\Omega \text{div} F dA = \int_{\partial\Omega} F \cdot \nu ds$$

Alternatively, we have, letting it be (2),

$$\int_\Omega u_x v dA = -\int_\Omega u v_x dA + \int_{\partial\Omega} u v \nu_x dA$$

In (1) we replace u_x by $(uv)_x = u_x v + u v_x$. Then (2) would be

$$\int_\Omega \text{div} F v dA = -\int_\Omega F \cdot \nabla v dA + \int_{\partial\Omega} v F \cdot \nu dS$$

Write φ as a Fourier series

$$\varphi(\theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \Big(A_n \cos(n\theta) + B_n \sin(n\theta) \Big)$$

$$u = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \Big(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) \Big)$$

call this $\star$ and find coefficients

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(s)\cos(ns)ds$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(s)\sin(ns)dS$$

plug into $\star$ and we have

$$u(n,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(s)ds + \sum_{n=1}^{\infty}\frac{r^n}{\pi}\int_{-\pi}^{\pi}\phi(s)\Big[\cos(ns)\cos(n\theta)+\sin(ns)\sin(n\theta)\Big]ds$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(s)\Big[1+2\Big(\sum_{n=1}^{\infty} r^n\cos n(\theta-s)\Big)\Big]ds$$

and then we have

$$\begin{aligned} 1+2\textstyle\sum_1^\infty r^n\cos(n(\theta-s)) &= 1+\textstyle\sum_1^\infty w^n+\overline{w}^n \\ &= 1+\frac{w}{1-w}+\frac{\overline{w}}{1-\overline{w}} \\ &= 1+\frac{w+\overline{w}-2|w|^2}{1-(w+\overline{w})+|w|^2} \\ &= \frac{1-|w|^2}{1-(w+\overline{w})+|w|^2} \end{aligned}$$

then we conclude

$$\frac{1}{2\pi}\Big[1+2\sum_{1}^{\infty} r^n\cos(n(\theta-s))\Big] = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta-s)+r^2}$$

which is the Poisson kernel, $P(r,\theta-s)$. Then for the problem

$$\triangle u = 0 \text{ in } B_1, u=\varphi(\theta) \text{ on } \partial B_1$$

we have

$$u(r,\theta) = \int_{-\pi}^{\pi}\varphi(s)P(n,\theta-s)ds = \varphi\star P(r,\cdot)$$

which is the convolution of two terms. This is

$$P(r,\theta-s) = \frac{1}{2\pi}\frac{1-n^2}{d^2}$$

Proposition 7.2.1. *Properties of Poisson Kernel. Written in polar coordinates, we have*

$$P(n,\theta) = \frac{1}{2\pi}\frac{1-n^2}{1-2n\cos\theta+n^2}$$

$$P(x_0,x) = \frac{1}{2\pi}\frac{1-|x_0|^2}{|x-x_0|^2}, x\in\partial B_1, x_0\in B_1$$

Then

$$u(x_0) = \int_{\partial B_1} P(x_0,x)\varphi(x)dx$$

Proposition 7.2.2. *Properties of the Poisson kernel: (1) $P>0$*
(2) $\int_{\partial B_1} P(x_0,x)dx = 1$
(3) $P(x_0,x)\to 0$ as $x_0\to\partial B_1$ and $|x-x_0|\geq\delta$.

Conclusion is that $P(x_0,x)\to\delta_{y_0}$ as $x_0\to y_0\in\partial B_1$ while here the kernel converges to Dirac delta.

7.3 More Dirichlet Problems

Recall we dealt with Dirichlet problem

$$\triangle u = 0, \text{ in } B = \{x = (x_1, x_2)/x_1^2 + x_2^2 < 1\}$$

while B is a unit disk. Then we know the solution follow

$$r^n \sin n\theta, r^n \cos n\theta$$

then we write solution for φ,

$$\varphi(\theta) = \frac{1}{2}A_0 + \sum_{m=1}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta)$$

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n \big(A_n \cos(n\theta) + B_n \sin(n\theta)\big)$$

and hence solve for general solution

$$u(r,\theta) = \int_0^{2\pi} \underbrace{\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta-s)+r^2}\varphi(s)\,d\theta}_{P(r,\theta-s),\text{ Poisson Kernel}}$$

using initial condition we have BE AWARE! IN EXAM!

$$u(x_0) = \int_{\partial B_1} \frac{1}{2\pi}\frac{1-|x_0|^2}{|x-x_0|^2}\varphi(x)dx$$

which we recommend to use since it works for high dimensions as well. This is like a simulated Brownian walk in a unit disk. Then we have

$$u(x_0) = \mathbb{E}\bigg(\varphi(x(r))|x(t) \text{ random walk, } J \text{ the exit time}\bigg)$$

which is a probabilistic statement for the general solution. In the reality, this really represents $P(x_0, x)dx$ is the probability density on ∂B_1 for the random walk that starts at x_0 to exit at $x \in \partial B_1$.

Now the question is what is the formula for B_a instead of B_1?

$$\triangle u = 0, \text{ in } B_a, u = \varphi \text{ on } \partial B_a$$

then we have $\tilde{u}(x/a) = u(x), \tilde{x} = \frac{x}{a} \in B_1$ for $x \in B_a$.

$$u(x_0) = \int_{\partial B_a} \left(\frac{x_0}{a}, \frac{x}{a}\right)\varphi(x)d\frac{x}{a}$$

instead of integrating over B_1 since we are now at $d\frac{x}{a}$. Note that

$$P(\frac{x_0}{a}, \frac{x}{a}) = \frac{1}{2\pi}\frac{1-|\frac{x_0}{a}|^2}{|\frac{x}{a}-\frac{x_0}{a}|^2} = \frac{1}{2\pi}\frac{a^2-|x_0|^2}{|x-x_0|^2}$$

so that

$$u(x_0) = \int_{\partial B_a} \frac{1}{2\pi}\frac{a^2-|x_0|^2}{|x-x_0|^2}\varphi(x)dx$$

and write this as $\star$.

7.4 Mean Value Property

In the formula $\star$ above, we take $x_0 = 0$. Then $u(0) = \int_{\partial B_a} \frac{1}{2\pi a} \varphi(x)dx$, and so $u(0) = \fint_{\partial B_a} u(x)dx$.

Proposition 7.4.1. *The mean value property states:*

$$u(x_0) = \fint_{\partial B_a(x_0)} u(x)dx$$

and $\triangle u = 0$ in Ω and $B_a(x_0) \subset \Omega$. Then

$$u(x_0) = \fint_{\partial B_a(x)} u(x)dx$$

Thus,

$$u \text{ as } MVP \Leftrightarrow \triangle u = 0$$

Theorem 7.4.2. *Strong Maximum Principle. If $\triangle u = 0$ in Ω and $\max u$ is achieved at the interior point, then u must be constant.*

Theorem 7.4.3. *Harmonic functions are infinitely differentiable in Ω. Consider $u \in C^\infty(\Omega)$, which is a consequence of $\star$. We can differentiate infinitely many times with respect to x_0. Moreover we have $u \in C^\infty$ and there are no corners inside Ω.*

See page 169 in text [1].

Now we can move on to discuss exteriors of circles. That is, we are looking at a full plane and move away a ball $r = \mathbb{R}^2 - B_1 = \{x_1^2 + x_2^2 > 1\}$. Can we find a harmonic function outside of the domain that satisfy boundary condition? In other words, we are trying to solve

$$\triangle u = 0 \text{ in } \Omega, u = \varphi \text{ on } \partial B_1, \text{ and } u(x) \to 0 \text{ as } |x| \to \infty.$$

The way we deal with infinite domain is delicate since we need to impose some sort of boundary condition at infinity. Consider $r \cos\theta$. There is a problem since the value does not tend to ∞ as we increase r. This is problematic. Let us consider Fourier series

$$\varphi(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \Big(A_n \cos(n\theta) + B_n \sin(n\theta) \Big)$$

and then

$$\varphi(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n} \Big(A_n \cos(n\theta) + B_n \sin(n\theta) \Big)$$

However, consider $u(x) = g(r)$, then

$$\triangle u = g'' + \frac{1}{r} g'(r)$$

while this is in two-dimensional and in general form we would have coefficient $\frac{n-1}{n}$. Then we have

$$g'' + \frac{1}{r} g' = 0, \text{ while } (rg')' = 0$$

and we get

$$rg'' + g' = 0 \Rightarrow rg' = A, \text{ while } A \text{ a constant}$$

which means $g' = \frac{A}{r}$. This implies that

$$g(r) = A \ln r + B$$

Consider Harmonic function $\triangle u = 0$ in $\Omega \subset \mathbb{R}^2$. Then we have

(1) Maximum principle (says that max and min of u occurs on $\partial\Omega$) and more we have strong max principle;

(2) Uniqueness of the Dirichlet Problem, that is,

$$\triangle u = 0 \text{ in } \Omega, u = \varphi \text{ on } \partial\Omega$$

(3) Mean Value Property.

$$u(x_0) = \fint u dx \text{ if } B_n(x_0) \subset \Omega$$

which can be $\frac{1}{2\pi r}\int_{\partial B_n(x_0)} u dx$.

(4) Solving Dirichlet Problem when $\Omega = B_a$. That is, we want $r^n \cos n\theta, r^n \sin n\theta, \ldots$. Then we have

$$\varphi = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big(A_n \cos n\theta + B_n \sin n\theta\Big)$$

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big(A_n r^n \cos n\theta + B_n r^n \sin n\theta\Big)$$

then consider condition

$$\varphi = \sin\theta - \cos 2\theta$$

then we have

$$u(t) = r\sin\theta - r^2\cos 2\theta$$

Recall the same procedure

$$r^{-n}\cos n\theta, r^{-n}\sin n\theta$$

and we have

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big(A_n r^{-n} \cos n\theta + B_n r^{-n} \sin n\theta\Big)$$

(5) Annulus. Now consider $r = B_a - B_b$ which is a ring (larger circle minus smaller circle). That is,

$$u = \varphi \text{ on } \partial B_a, u = \psi \text{ on } \partial B_b$$

and then we get

$$g(n)h(\theta) = \begin{cases} 1, & \log n \\ n^{\pm 1}\cos\theta, & n^{\pm 1}\sin\theta \\ n^{\pm 2}, & n^{\pm 1}\sin 2\theta \end{cases}$$

and we say this is harmonic and solve for

$$\frac{-h''}{h} = \lambda, r^2 g'' + r g' - \lambda g = 0$$

then this gives us

$$u = A_0 + B_0 \ln r + \sum_{n=1}^{\infty}\Big[(A_n r^n + B_n r^{-n})\cos n\theta + (C_n r^n + D_n r^{-n})\sin n\theta\Big]$$

call this $\star$ and we find A_n, B_n, C_n, D_n from the Fourier coefficients of φ when $r = a$ and ψ when $r = h$.

Example 7.4.4. Consider $a = 2$, $h = 1$, $\Omega = B_2 - B_1$. Consider $\varphi = 3 - 2\sin\theta + 7\cos 2\theta$, and $\psi = \frac{1}{2} - 3\cos\theta$ as the setup.

$$r = 2 \Rightarrow A_0 + B_0 \ln 2 = 3$$

$$r = 1 \Rightarrow A_0 + B_0 \ln 1 = A_0 = \frac{1}{2}$$

then you can solve for A_0 and B_0,

$$A_0 = \frac{1}{2}, B_0 \frac{5}{2}\frac{1}{\ln 2}$$

Next, we move forward to find

$$r = 2 \Rightarrow A_1 2 + B_1 2^{-1} = 0, C_1 2 + D_1 2^{-1} = -2$$

$$r = 1 \Rightarrow A_1 + B_1 = -3, C_1 + D_1 = 0$$

and we solve for A_1, B_1 and C_1, D_1.

(6) Ω is a rectangle with $\Omega = [a,b] \times [c,d]$. Also consider

$$\triangle u = 0 \text{ in } \Omega, u = \varphi \text{ on } \partial\Omega$$

and consider changing of variables $g(x)h(y)$ and we have

$$\frac{-g''}{g} = \lambda, \frac{-h''}{h} = -\lambda$$

If $\lambda \geq 0$, then we have

$$\sin\sqrt{\lambda}x e^{\pm\sqrt{\lambda}y}, \cos\sqrt{\lambda}x e^{\pm\sqrt{\lambda}y}$$

and consider initial data

$$x \in [0,\pi] \times y \in [0,1]$$
$$u = 0 \text{ when } x = 0, x = \pi$$
$$u = \varphi \text{ when } y = 0$$
$$u = \psi \text{ when } y = 1$$

we are essentially looking for solution of

$$u(x,y) = \sum_{n=1}^{\infty}\left(A_n e^{ny} + B_n e^{-ny}\right)\sin(nx)$$

We find A_n and B_n from the Fourier sine series of φ when $y = 0$ and ψ when $y = 0$. Then we form setup

$$\varphi = \sum C_n \sin(nx), \psi = \sum D_n \sin(nx)$$

(7) The Dirichlet Principle.

Theorem 7.4.5. *A harmonic function u minimizes the energy*

$$E(w) = \frac{1}{2}\int_{\Omega} |\nabla w|^2 dx$$

among all other functions which have the same boundary data on $\partial\Omega$ as u.

Proof. Assume that $\triangle u = 0$ in Ω. Consider another function v with the same boundary data as u. Then

$$\mathbb{E}(v) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx = \frac{1}{2}\int |\nabla(v-u) + \nabla u|^2 dx$$

then we solve

$$\begin{array}{rl} & \frac{1}{2}\int_\Omega |\nabla(v-u)|^2 + 2\nabla(v-u)\cdot\nabla u + |\nabla u|^2 dx \\ = & \mathbb{E}(v-u) + \mathbb{E}(u) + \int_\Omega \nabla(v-u)\nabla u dx \\ & -\int (v-u)\text{div}(\nabla u)dx + \int (v-u)(\nabla u)\nu ds \end{array}$$

and the conclusion is if $v = u$ on $\partial\Omega$ and $\triangle u = 0$ then

$$E(v) = E(v-u) + E(u) \geq E(u)$$

□

7.5 Dirichlet Principle

Consider Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$. That is, we may define $x \in \omega$ such that $x = (X_1, X_2)$ or $x = (x_1, x_2, x_3)$. Then we define w in Ω as

$$E(W) = \frac{1}{2}\int_\Omega |\nabla w|^2 dx$$

The fact is that if $\triangle u = 0$ then $E(v) \geq E(u)$ if $v = u$ on $\partial\Omega$. $E(u)$ is the minimum energy among all functions with the same boundary data s u. From last discussion,

$$E(v) = E(u) + E(v-u) + \underbrace{\int_\Omega \nabla u \cdot \nabla(v-u)dx}_{(1)}$$

Then by Divergence Theorem and integration by parts, we have

$$\int_\Omega \text{div}F h dx = -\int_\Omega F\cdot\nabla h y + \int_{\partial\Omega} hF\cdot\nu dS$$

so we have

$$1) = -\int_\Omega \text{div}(\nabla u)(v-u)dx + \int_{\partial r}(v-u)u\nu dS$$

while on the boundary $v - u = 0$ and $\text{div}F = 0$, which gives us (1) = 0. Then we conclude that

$$E(v) = E(u) + E(v-u)$$

to describe the energy and hence we know

$$E(v) = E(u) + E(v-u) \geq E(u)$$

Assume $E(u)$ is the min for

$$E(u) = \min\{E(w) | w = u \text{ on } \partial\Omega\}$$

and we have

$$E(u+t\varphi) \text{ while } \begin{cases} t & \text{all} \\ \varphi = 0 & \text{on } \partial r \end{cases}$$

which gives us

$$E(u+t\varphi) = \frac{1}{2}\int_\Omega |\nabla u + t\nabla\varphi|^2 dx = \frac{1}{2}\int_\Omega |\nabla u|^2 + 2t\nabla u\nabla\varphi + t^2|\nabla\varphi|^2 dx$$

and this solves for

$$\lim_{t\to 0}\frac{E(u+t\varphi)-E(u)}{t}=\int_\Omega \nabla u\nabla\varphi dx=0$$

which leads us to conclude

$$0=\int_\Omega \nabla u\nabla\varphi dx=-\int_\Omega \mathrm{div}(\nabla u)\varphi dx+\int_{\partial\Omega}\varphi u_\nu dS$$

and since $\varphi = 0$ the conclusion is

$$0=\int_\Omega (triangleu)\varphi dx$$

for all function φ such that $\varphi = 0$ on the boundary $\partial\Omega$, that is, the Laplace is zero,

$$\triangle u = 0$$

which is called Euler-Lagrange Equation for Energy E.

Now an interesting question is: what happens if Good Example!

$$F(w)=\int_\Omega \frac{1}{2}|\nabla w|^2+5w^2dx$$

then

$$0=\lim_{t\to 0}\frac{E(u+t\varphi)-E(u)}{t}=\int_\Omega \nabla u\nabla\varphi+5\cdot 2u\varphi)dx=\int_\Omega(-\triangle u)\varphi+10u\varphi dx$$

Euler-Lagrange equation for F is

$$-\triangle u+10u=0$$

$$A(w)=\int_\Omega\sqrt{1+|\nabla w|^2}dx=\int_\Omega G(\nabla w)dx$$

so we get

$$0=\int_\Omega \nabla G(\nabla u)\nabla\varphi dx$$

using integrating by parts we get E-L equation

$$\mathrm{div}(\nabla G(\nabla u))=0$$

which is a non-linear equation and we cannot build solutions out of solutions. Point is that if we are given some energy equation we can solve that and simplify the form into something simpler (sometimes using Divergence Theorem and integrating by parts).

8 GREEN'S FUNCTION

We discuss Green's functions and Laplace equation on general domain. Consider any function, say

$$\int_\Omega \triangle u dx=\int_\Omega u_\nu dS$$

with the left side as the $\mathrm{div}(\nabla u)$, the divergence of u. If $\triangle u = 0$, then

$$\int_{\partial\Omega} u_\nu dS=0$$

and then

$$\int_{B_r} \triangle u dx = \int_{\partial B_a} u_\nu dS = 0 \Rightarrow \fint_{\partial B_r} u_\nu dS = 0$$

and we can conclude that

$$\frac{d}{d\Omega}\left(\fint_{\partial B_r} udS\right) = 0$$

which leads to conclusion

$$\fint_{\partial B_r} udS = \text{const in } r$$

and that constant would be u evaluated at 0, i.e. $u(0)$.

Theorem 8.0.1. *Mean Value Property. states that*

$$u(x_0) = \fint_{\partial B_r(x_0)} udS$$

which means that

$$\triangle u = 0 \Rightarrow u(x_0) = \fint_{\partial B_r(x_0)} udS$$

or else

$$\triangle u \geq 0 \Rightarrow u(x_0) \leq \fint_{\partial B_r(x_0)} udS$$

or

$$\triangle \geq 0 \Rightarrow u(x_0) \geq \fint_{\partial B_r(x_0)} udS$$

Which are the radial functions that are harmonic? Then consider

$$u(x) = f(|x|) = f(r), \triangle u = f'' + \frac{(n-1)}{r} f' = 0$$

and then

$$\frac{f''}{f'} + \frac{n-1}{n} = 0$$

$$(\ln f' + (n-1)\ln r)' = 0$$

taking integral and we have

$$\ln f' + (n-1)\ln r = C$$

then solve for $f' = cr^{1-n}$ and we have

$$f = \begin{cases} Ar^{2-n} + B & n = 3, 4, ... \\ A\ln r + B & n = 2 \end{cases}$$

while in three-dimension we have

$$0 = \int_{B_1} \triangle(|x|^{-1})dx = \int_{\partial B_1} \frac{d}{dr} r^{-1} dS = -4\pi$$

Then we get

$$\triangle(\frac{1}{4\pi|x|}) = \delta_0$$

which comes from the tip of the point. Thus,

$$0 = \int_{B_1 - B_\epsilon} \triangle(|x|^{-1})dx = \int_{\partial B_1} \frac{d}{dr}(r^{-1})dS - \int_{\partial B_\epsilon} \frac{d}{dr}(r^{-1})dS$$

while the first term is -4π and the second is the same as the first as well, which is independent of ϵ.

8.1 Fundamental Solution

From last time, we conclude fundamental solution

$$\Gamma(x) = \frac{1}{4\pi}\frac{1}{|x|}$$

and we have

$$-\triangle\Gamma = \delta_0$$

which is the fundamental solution in 3-dimensional Laplace equation. Then we have

$$\Gamma = \frac{1}{4\pi}\frac{1}{r} \text{ while } r = |x|$$

Then $\Gamma(x)$ can be

$$-\frac{1}{2\pi}\ln|x|$$

with

$$-\triangle\Gamma = \delta_0$$

as the fundamental solution in two-dimensional Laplace equation. Thus, we have

$$\int_\Omega v\triangle u dx = -\int_\Omega \nabla v\cdot\nabla u + \int_\Omega v\underbrace{\nabla u_\nu}_{\nabla u\nu} dS$$

which gives us Green's first identity. Write the same identity by switching u and ν and subtract and we get

$$\int_\Omega v\triangle u - u\triangle v dx = \int_\Omega vu_\nu - uv_\nu dS$$

which gives us Green's second identity.

In the second identity, assuming $\triangle u = 0$, we plug $v = \Gamma(x-x_0) = \Gamma_{x_0}$ and $\triangle u = \triangle\Gamma_{x_0} = -\delta_{x_0}$ so that

$$\int_\Omega \Gamma_{x_0}\underbrace{\triangle u}_{=0} - u\underbrace{\triangle\Gamma_{x_0}}_{-\delta x_0} dx = \int_{\partial\Gamma}\Gamma_{x_0}u_\nu - (\Gamma_{x_0})_\nu u dS$$

and thus the general function at x_0 would be

$$u(x_0) = \int_{\partial\Omega} u_\nu\Gamma_{x_0} - u(\Gamma_{x_0})_\nu dS$$

with representation graph to be a field of Ω with a point x_0 in the middle.

8.2 Green's Function Solving Dirichlet Problem

Consider

$$\triangle u = 0 \text{ in } \Omega, \text{ and } u = \varphi \text{ on } \partial\Omega$$

In representation formula μ_ν on ∂r is not known. We work with a modified Green's function of Γ_{x_0} and

$$G(x, x_0) = \Gamma_{x_0} - h$$

with h such that

$$\triangle h = 0 \text{ in } \Omega, h = \Gamma_{x_0} \text{ on } \partial\Omega$$

then we have

$$-\triangle G(x, x_0) = \delta_{x_0} \ (1)$$

$$G(x, x_0) = 0 \text{ on } \partial\Omega \ (2)$$

Now we plug $G(x,x_0)$ (instead of Γ_{x_0}) into Green's second identity. Then we have

$$u(x_0) = \int_{\partial\Omega} -u(G(x,x_0))_\nu dS$$

then we have solution to the Dirichlet problem

$$u(x_0) = -\int_{\partial\Omega} \varphi(G_{x_0})_\nu dS$$

Example 8.2.1. Consider Ω is a half space and $\Omega = \{x \in \mathbb{R}^3 \text{ with } x_3 > 0\}$. Then

$$\Gamma_{x_0} \frac{1}{4\pi} \frac{1}{|x - x_0|}$$

We have h in set up $\star$ to the $\Gamma_{\bar{x}_0}$ where $\bar{x}_0 = ((x_0)_1, (x_0)_2, -(x_0)_3)$ then

$$G(x,x_0) = \Gamma_{x_0} - \Gamma_{\bar{x}_0} = \frac{1}{4\pi}\left(\frac{1}{|x-x_0|} - \frac{1}{|x-\bar{x}_0|}\right)$$

To solve the general Dirichlet problem, we need to compute $-(G(x,x_0))_\nu$ on the boundary $\partial\Omega$. In the case of half-plane (page 191 in text [1]), this is to compute

$$-(G(x,x_0))_\nu = \vec{e}_3 \cdot \nabla G(x,x_0)$$

and then

$$\nabla\Gamma = \frac{1}{4\pi}\frac{1}{|x|^2}\frac{x}{|x|}$$

and

$$\nabla\Gamma_{x_0} = \frac{1}{4\pi}\frac{x_0 - x}{|x_0 - x|^3}$$

$$\begin{aligned}
\nabla G(x,x_0)\cdot e_3 &= \nabla\Gamma_{x_0}\cdot e_3 - \nabla\Gamma_{\bar{x}_0}\cdot e_3 \\
&= \frac{1}{4\pi}\frac{(x_0 - x)\cdot e_3}{|x-x_0|^3} - \frac{1}{4\pi}\frac{(\bar{x}_0 - x)\cdot e_3}{|x-\bar{x}_0|^3} \\
&= \frac{1}{4\pi}\frac{x_0\cdot e_3 - \bar{x}_0\cdot e_3}{|x-x_0|^3} \\
&= \frac{1}{2\pi}\frac{x_0\cdot e_3}{|x-x_0|^3}
\end{aligned}$$

which gives us the solution for

$$u(x_0) = \int_{\partial\Omega} \varphi(x)\frac{(x_0)_3}{2\pi|x-x_0|^3}dx$$

8.3 Half-space in Sphere

For another situation, consider $\Omega = B_1 = \{x \in \mathbb{R}^3 \setminus |x| < 1\}$ and we consider

$$\triangle h = 0 \text{ in } B_1, h = \Gamma_{x_0} \text{ on } \partial B_1$$

then

$$x_0^\star = \frac{x_0}{|x_0|^2}$$

say you take a point $|x| = 1$ we compute

$$\begin{aligned}
|x - x_0^\star|^2 &= xx - 2xx_0^\star + |x_0^\star|^2 \\
&= 1 - 2\frac{x - x_0}{|x_0|^2} + \frac{1}{|x_0|^2} \\
&= \frac{1}{|x_0|^2}\left(|x_0|^2 - 2xx_0 + 1\right) \\
&= \frac{1}{|x_0|^2}|x - x_0|^2
\end{aligned}$$

and we conclude that

$$|x - x_0^\star| = \frac{1}{x_0}|x - x_0|$$

then

$$G(x, x_0) = \Gamma_{x_0} - |x_0|\Gamma_{x_0^\star} = 0 \text{ on } \partial B_1$$

Index

binomial distribution, 21
Burgers' equation, 5

characteristic ODE, 4
completeness, 30
convergence in mean square, 27

d'Alembert equation, 11
Dirichlet Principle, 46
Dirichlet Problem, 45
Dirichlet problem, 39

elastic homogeneous string, 6
Euler-Lagrange Equation for Energy, 48

Green's first identity, 50
Green's second identity, 50

harmonic functions, 34
heat equation, 16
holomorphic, 36

independent variables, 4

Kardar - Parisi - Zhang equation, 5

Laplace equation, 9

Maximum principle, 45
Mean Square Convergence, 32
Mean Value Property, 49
mean value property, 44

Neumann problem, 40
Newmann Problem, 9

Point-wise Convergence, 32
Point-wise convergence, 27
Poisson Kernel, 42

Random Walk, 21
Rankine–Hugoniot conditions, 9
Riemann-Lebesgue lemma, 30, 32
Riemann-Lebesgue-Fourier, 28
Robin conditions, 24
rotational-dilation matrix, 36

Schrodinger equation, 21
Strong Maximum Principle, 44

transport equation, 4
transverse vibrations, 6

Uniform Convergence, 32
Uniform convergence, 27
Uniqueness of the Dirichlet Problem, 45

vibrating Drumhead, 7

wave equation, 9
wave equations, 25

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[1] Strauss, A. Walter, "Partial Differential Equation".